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STRONG TUTTE FUNCTIONS OF MATROIDS AND GRAPHS

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ABSTRACT. A strong Tutte function of matroids is a function of finite matroids which satisfies $F(M_1 \oplus M_2) = F(M_1)F(M_2)$ and $F(M) = a_e F(M \backslash e) + b_e F(M/e)$ for e not a loop or coloop of M, where a_e, b_e are scalar parameters depending only on e. We classify strong Tutte functions of all matroids into seven types, generalizing Brylawski's classification of Tutte-Grothendieck invariants. One type is, like Tutte-Grothendieck invariants, an evaluation of a rank polynomial; all types are given by a Tutte polynomial. The classification remains valid if the domain is any minor-closed class of matroids containing all three-point matroids. Similar classifications hold for strong Tutte functions of colored matroids, where the parameters depend on the color of e, and for strong Tutte functions of graphs and edge-colored graphs whose values do not depend on the attachments of loops. The latter classification implies new characterizations of Kauffman's bracket polynomials of signed graphs and link diagrams.

INTRODUCTION

A *Tutte-Grothendieck invariant* of matroids is a function F from (finite) matroids to a domain of scalars, which satisfies the multiplicative, additive, and invariance laws:

$$F(M) = F(M_1)F(M_2) \quad \text{if } M = M_1 \oplus M_2, \tag{M}$$

$$F(M) = F(M \backslash e) + F(M/e) \tag{A}$$

if e is a nonseparating point of M (that is, neither a loop nor coloop), and

$$F(M_1) = F(M_2) \quad \text{if } M_1 \cong M_2. \tag{I}$$

Brylawski in [1] proved all such functions to be the evaluations of a certain two-variable polynomial function of matroids known as the Tutte polynomial, $t_M(x, y)$, or equivalently (as Crapo had shown in [2]) of the rank polynomial (or "rank-generating polynomial") $R_M(u, v)$, whose definition is very different but which equals $t_M(u+1, v+1)$. These results were extensions to matroids of seminal ideas introduced originally for graphs by Tutte [11, 12].

Research into polynomial invariants of knots by Thistlethwaite (especially [9]) and Kauffman led the latter to define a version of the Tutte polynomial for

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graphs with edges labeled $+1$ and -1 (announced in [4] with details in [5, 6]). Kauffman's "Tutte polynomial of a signed graph" obeys laws slightly different from (M, A, I); for instance, additivity is replaced by a parametrized law

$$F(\Gamma) = aF(\Gamma \backslash e) + bF(\Gamma / e),$$

where $(a, b) = (B, A)$ or (A, B) depending on the label of edge e. (Here A and B are indeterminates.) I was inspired by Kauffman's polynomial to look for a generalization of Brylawski's classification theorem to functions of colored matroids and even more generally to functions of matroids which need satisfy only (M) and the linearity property

$$\text{(L)} \qquad F(M) = a_e F(M \backslash e) + b_e F(M/e)$$

if e is a nonseparating point of M, where a_e and b_e are arbitrary scalar parameters depending only on the element e. I call these *strong Tutte functions* of matroids. The main results of this paper are a classification of strong Tutte functions of all matroids and of analogous functions of colored matroids, where the parameters depend on the color of e and the function is invariant under color-preserving isomorphisms. (Colored matroids are the proper generalization of Kauffman's ± 1-labelled or "signed" graphs.)

The main theorems are stated in §2. Their most notable aspect is that there are not one universal strong Tutte function, as with Tutte-Grothendieck invariants, but seven. One type, called *normal*, is a parametrized analog of Tutte-Grothendieck invariants, being given both by a two-variable parametrized rank polynomial $R_M(\mathbf{a}, \mathbf{b}; u, v)$ (Example 2.1) and a parametrized Tutte polynomial (§7). This type exists for all choices of parameters, as do the nil Tutte functions, which are zero on all nonempty matroids (and which are not in general normal). Other abnormal types exist only for special choices of parameters. Every type has an expression by a parametrized Tutte polynomial, which expands it as a sum, over all bases, of certain monomials; but only the normal type has a rank polynomial and only that type depends substantially on the structure of the matroid, as one can see from the detailed descriptions in §2.

The remarkable feature of the proof is that it depends only on connected matroids of three points and their minors (contractions of their submatroids). Therefore the classification holds good for functions defined on any minor-closed class of matroids, with points in a universe U, which contains all three-point circuits and cocircuits: for instance, the class of planar-graphic matroids or that of transversal matroids. It is even possible to omit some of the circuits and cocircuits without getting new strong Tutte functions, but to prove this will usually require a more difficult analysis. (I plan to treat elsewhere Tutte functions whose domain is *principal*, that is, the class of all minors of a single fixed matroid.)

A synopsis of this article is as follows: We begin with precise definitions, some facts about domains of Tutte functions, and the statements of the main theorems. The proofs occupy §§3 to 6. We examine the parametrized Tutte polynomial in §7, scaling operations in §8, and duality and permutation in §9, including the important concept of self-conjugacy, or being invariant under the combination of duality and a permutation. We conclude with the application to graphs, which is surprisingly not quite automatic, and to Kauffman's Tutte polynomial, which turns out (not surprisingly) to depend only on the graphic matroid. Our theory yields new axioms for Kauffman's polynomial which seem slightly more natural than the original ones.

In a future work I plan to characterize *weak Tutte functions*, which need satisfy only (L), by finding the parametrized Tutte algebra of matroids, the analog of the Tutte-Grothendieck algebra developed by Brylawski in [1] (again based on the fundamental idea of Tutte in [11]).

1. Tutte functions and basic definitions

We follow the matroid notation and terminology of [13] with a few variations. All our matroids are finite. The point set of a matroid M is $E(M)$, but we often write $e \in M$ as shorthand for $e \in E(M)$ and $S \subseteq M$ for $S \subseteq E(M)$. The rank, nullity, and corank of $S \subseteq M$ are $\mathrm{rk}(S)$, $\mathrm{nul}(S) = |S| - \mathrm{rk}\, S$, and $\mathrm{cork}(S) = \mathrm{rk}\, M - \mathrm{rk}\, S$; its complement is $S^c = E(M)\backslash S$. We write $E_0(M)$ for the set of loops and $E_1(M)$ for that of coloops of M. The elements of $E_0(M) \cup E_1(M)$ are called *separating* points of M, and those of $E_*(M) = [E_0(M) \cup E_1(M)]^c$ are *nonseparating* elements. By $(A)_r$ we mean the uniform matroid on A of rank r; for example, $(e)_1$ is a coloop. The null or pointless matroid, on point set $\varnothing$, is written $\varnothing$. The dual matroid of M is $M^\perp$. A matroid is *discrete* if every point is a separator, that is, a loop or coloop. An n-point circuit matroid is C_n; a *digon* is a C_2, a *triangle* is a C_3, and a *triad* is a $C_3^\perp$. For $e, f \in M$, $e||f$ means e and f are a parallel pair and $e||^\perp f$ means they are a series pair. We say M is a matroid *on* a class U if $E(M) = U$, and *in* U if $E(M) \subseteq U$.

The script letter $\mathscr{M}$ will always denote a class of matroids. $\mathscr{M}$ is *in* U if $E(M) \subseteq U$ for every $M \in \mathscr{M}$ and *point-covering* if it is in U and $(e)_0, (e)_1 \in \mathscr{M}$ for all $e \in U$. It is *minor-closed* if every minor of a member of $\mathscr{M}$ is again in $\mathscr{M}$. By $\mathscr{M}_{(2)}$ we mean the class of matroids in $\mathscr{M}$ which are digons or have at most one point. By $\mathscr{M}_{(3)}$ we mean $\mathscr{M}_{(2)}$ together with the connected 3-point matroids (that is, the triangles and triads) in $\mathscr{M}$ and those of their minors which lie in $\mathscr{M}$.

We make heavy use of vectors in K^2 (where K is a field), which we regard as column vectors although often writing them horizontally in text for convenience's sake. Two such vectors are *parallel* or (*homogeneously*) *collinear*, $p||q$, if one is a scalar multiple of the other; this permits p or $q = 0$, which is slightly nonstandard. Vectors are *affinely collinear* if they lie on an affine line. If $p, p', \ldots \in K^2$, we write (p, p', p'') (for instance) for the matrix whose columns are p, p', and p'', and $|p, p'|$ for the determinant of (p, p'). Thus $p||q \Leftrightarrow |p, q| = 0$. We write $p \cdot q$ for the standard inner product; thus $p \perp q \Leftrightarrow p \cdot q = 0$.

A *function of matroids* is a function F from some class $\mathscr{M}$ of matroids in a class U (the *point universe of* F) into a field K. For instance $\mathscr{M}$ may be the class $\mathscr{M}(U)$ of all (finite) matroids in U; then F is called *global* (*over* U). Or it may be the class $\mathscr{M}(M)$ of all (finite) minors of a fixed matroid M; then F is *principal*. For a function F we always write

$$x_e = F((e)_1), \quad y_e = F((e)_0), \quad q_e = \begin{pmatrix} x_e \\ y_e \end{pmatrix} \quad \text{for } e \in U;$$

the former are the *point values* of F and the latter are its *point-value vectors*. (Some of them may be undefined, if the corresponding matroids are not in $\mathscr{M}$.)

We call $\mathbf{x} = (x_e : e \in U)$ and $\mathbf{y} = (y_e : e \in U)$ respectively the *coloop* and *loop value sequences* of F and $\mathbf{q} = (q_e : e \in U) = \binom{\mathbf{x}}{\mathbf{y}}$ the *point value sequence.* A function F is *nil* if $F(M) = \varnothing$ for every nonnull matroid M in its domain. We call F *degenerate* if $F(M)$ is determined by $E(M)$, $E_0(M)$, $E_1(M)$, and rk M.

A *strong Tutte function* F of matroids is a function F on a minor-closed point-covering class $\mathscr{M}$ in a class U, together with a pair of *parameters* a_e, $b_e \in K$ for each $e \in U$, which satisfies (M) and (L) for every matroid in its domain. We call $\mathbf{a} = (a_e : e \in U)$ and $\mathbf{b} = (b_e : b \in U)$ respectively the *deletion* and *contraction* parameter sequences of F, and we let $p_e = \binom{a_e}{b_e}$, $\mathbf{p} = \binom{\mathbf{a}}{\mathbf{b}}$. A *weak Tutte function* is the same, except it need not obey (M). A Tutte function (strong or weak) is thus a pair $(F, \mathbf{p})$ but we usually leave the parameters implicit. Tutte functions F_1 and F_2 are *equal* $(F_1 = F_2)$ if their domains and codomains are the same and $F_1(M) = F_2(M)$ for every M in the domain; they are *identical* $(F_1 \equiv F_2)$ if they also have the same parameter sequences, $\mathbf{p}_1 = \mathbf{p}_2$. Our usual point of view is that the domain, codomain, and parameters have been fixed in advance and we study the associated Tutte functions.

Evidently a strong Tutte function F is completely determined by its domain, codomain, parameters (in which its point universe is implicit), and point values, and $F(\varnothing)$ if F is nil (since $F(\varnothing) = 1$ otherwise). Thus we may write $F = F[\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y}] = F[\mathbf{p}; \mathbf{q}]$ to mean that F is a strong Tutte function with the indicated parameters and point values. The precise domain $\mathscr{M}$ is thereby unspecified but in fact $F[\mathbf{p}; \mathbf{q}]$ has a unique largest possible domain. Given $\mathbf{p}$ and $\mathbf{q}$, let $\mathscr{M}[\mathbf{p}; \mathbf{q}]$ consist of all matroids M in U for which there exists a strong Tutte function with domain $\mathscr{M}(M)$ and parameters and point values $\mathbf{p}|_{E(M)}$ and $\mathbf{q}|_{E(M)}$. We call $\mathscr{M}[\mathbf{p}; \mathbf{q}]$ the *natural domain* of a strong Tutte function $F = F[\mathbf{p}; \mathbf{q}]$, because of

Theorem 1.1. *Let* $\mathbf{p}, \mathbf{q} \in (K^U)^2$. *There is a strong Tutte function* $F[\mathbf{p}; \mathbf{q}]$ *whose domain is* $\mathscr{M}[\mathbf{p}; \mathbf{q}]$. *Any strong Tutte function* F_0 *of matroids in* $U_0 \subseteq U$ *whose parameter and point value sequences are* $\mathbf{p}\big|_{U_0}$ *and* $\mathbf{q}\big|_{U_0}$ *has domain contained in* $\mathscr{M}[\mathbf{p}; \mathbf{q}]$ *and extends to the strong Tutte function* $F[\mathbf{p}; \mathbf{q}]$ *on* $\mathscr{M}[\mathbf{p}; \mathbf{q}]$. □

We say a strong Tutte function $F = F[\mathbf{p}; \mathbf{q}]$ on an arbitrary domain has *global type* if $\mathscr{M}[\mathbf{p}; \mathbf{q}] = \mathscr{M}(U)$. If the domain is clear from context we may, loosely, call F "global."

Given a parameter sequence $\mathbf{p} \in (K^U)^2$, let $U^* = \{e \in U : p_e \neq 0\}$.

Lemma 1.2. *A strong Tutte function* $F = F[\mathbf{p}; \mathbf{q}]$ *is degenerate if* $|U^*| \leq 3$.

Proof. Writing $E_* = E_*(M)$, etc., we have in general

$$F(M) = F(M|E_*) \prod_{e \in E_1} x_e \prod_{e \in E_0} y_e.$$

Supposing $|E_*| \leq 3$, $M|E_*$ is determined by E_* and $\mathrm{rk}(M|E_*) = \mathrm{rk}\ M - |E_1|$, hence by the information available to a degenerate function of M. Hence if $E_* \subseteq U^*$ and $|U^*| \leq 3$, M is completely determined by that information.

Since linearity forces $F(M) = 0$ if $E_* \not\subseteq U^*$, it follows that F is degenerate. □

2. The main theorems

We begin by listing the seven kinds of global strong Tutte functions, giving for each type not just its parameters and point values but also a general formula. Every kind is essential: none is contained by the rest.

Example 2.1 (Normal functions). Given fixed $\mathbf{a}, \mathbf{b} \in K^U$, the *parametrized rank polynomial* of a matroid M in U is the two-variable polynomial

$$R_M(\mathbf{a}, \mathbf{b}; u, v) = \sum_{S \subseteq E(M)} u^{\text{cork } S} v^{\text{nul } S} \cdot \prod_{e \in S^c} a_e \cdot \prod_{e \in S} b_e.$$

For the empty matroid this equals 1. A function of matroids which has the form $F(M) = R_M(\mathbf{a}, \mathbf{b}; u, v)$ for suitable $\mathbf{a}, \mathbf{b} \in K^U$ and $u, v \in K$ is called *normal.* Any other function is *abnormal.* Routine calculations show that normal functions are strong Tutte functions. They are, as we shall see, the only significant ones which exist for all parameters, and the only nondegenerate global ones.

Observe that

$$R_{M^\perp}(\mathbf{a}, \mathbf{b}; u, v) = R_M(\mathbf{b}, \mathbf{a}; v, u),$$

that a normal function has $x_e = a_e u + b_e$, $y_e = a_e + b_e v$, or more concisely

$$q_e = \begin{pmatrix} u & 1 \\ 1 & v \end{pmatrix} p_e \quad \text{for } e \in U, \tag{2.1}$$

and that the usual matroid rank polynomial [2] is $R_M(u, v) = R_M(\mathbf{1}, \mathbf{1}; u, v)$.

A parametrized rank polynomial seems not to have been defined for matroids before, but analogs for graphs have appeared in the literature (see §10).

Example 2.2 (Nil functions). A nil function is a strong Tutte function if and only if $F(\varnothing)$ is idempotent, that is, 1 or 0. It is normal if $F(\varnothing) = 1$ and all p_e are collinear—hence in particular for Tutte-Grothendieck invariants, where all $p_e = (1, 1)$—but not otherwise.

Example 2.3 (Elementary functions). Here we have two dual examples. Let $E_0 = E_0(M)$ and $E_1 = E_1(M)$. Given a fixed $\mathbf{a} \in K^U$ and arbitrary $u \in K$ and $\mathbf{y} \in K^U$, the (*parametrized*) *rank-loop polynomial* of M is

$$u^{\text{rk } M} \prod_{e \in M \setminus E_0} a_e \cdot \prod_{e \in E_0} y_e = R_{M \setminus E_0}(\mathbf{a}, \mathbf{0}; u, v) \prod_{e \in E_0} y_e.$$

For the empty matroid this equals 1. We also define the (*parametrized*) *corank-coloop polynomial*,

$$v^{\text{nul } M} \prod_{e \in M \setminus E_1} b_e \cdot \prod_{e \in E_1} x_e = R_{M \setminus E_1}(\mathbf{0}, \mathbf{b}; u, v) \prod_{e \in E_1} x_e,$$

where $\mathbf{b} \in K^U$ is fixed and $v \in K$ and $\mathbf{x} \in K^U$ are arbitrary.

A function which can be expressed as the rank-loop polynomial for suitable arguments is called *primal elementary.* One which equals the corank-coloop polynomial is *dual elementary.* A primal (or dual) elementary function is a strong Tutte function with parameters $\mathbf{a}$ and $\mathbf{0}$ (or, $\mathbf{0}$ and $\mathbf{b}$). For a primal elementary function F we have

$$x_e = a_e u, \qquad y_e \text{ is arbitrary.} \tag{2.2a}$$

For a dual elementary function,

(2.2b) $$x_e \text{ is arbitrary}, \quad y_e = b_e v.$$

Example 2.4 (Pairlike functions). A strong Tutte function is *pairlike* if $F(\varnothing) = 1$ and $p_e = q_e = 0$ for all but at most two points. It is *strictly pairlike* if $p_e \neq 0$ for exactly two points.

Example 2.4A (Planar pairlike functions). Distinguish two points $d_1, d_2 \in U$. Choose fixed $p_{d_1}, p_{d_2} \in K^2$ and set $p_e = 0$ for $e \in U \backslash \{d_1, d_2\}$. Let $u, v, w \in K$ be arbitrary and define a function G on $\mathscr{M}(U)$ by

$$G(\varnothing) = 1,$$
$$q_e(G) = \begin{pmatrix} u & w \\ w & v \end{pmatrix} p_e \quad \text{for } e \in U,$$
$$G((d_1, d_2)_1) = p_{d_1}^T \begin{pmatrix} u & w \\ w & v \end{pmatrix} p_{d_2} = a_{d_1}a_{d_2}u + (a_{d_1}b_{d_2} + b_{d_1}a_{d_2})w + b_{d_1}b_{d_2}v,$$
$$G((d_1)_i \oplus (d_2)_j) = G((d_1)_i)G((d_2)_j) \quad \text{for } i, j = 0, 1,$$
$$G(M) = 0 \quad \text{if } E(M) \not\subseteq \{d_1, d_2\}.$$

A function of this form is called *planar pairlike*; if p_{d_1} and p_{d_2} are linearly independent it is *truly-planar pairlike*. It is clearly a strong Tutte function.

If p_{d_1} and p_{d_2} are collinear but nonzero, this falls under Example 2.4B. If one or both p_{d_i} are zero, it falls under Example 2.5B or A.

Example 2.4B (Collinear pairlike functions). Here again there are two special points $d_1, d_2 \in U$. Let $p = (a, b) \in K^2$ be any nonzero vector, let $p^\perp = (b, -a)$, and choose $\boldsymbol{\pi} \in K^U$ so that $\pi_e = 0$ if $e \neq d_1, d_2$. Set $p_e = \pi_e p$. Also let $r = (r_1, r_0) \in K^2$ be a vector not orthogonal to p. Let $t, z_{d_1}, z_{d_2} \in K$ be arbitrary and define a function H on $\mathscr{M}(U)$ by

$$H(\varnothing) = 1,$$
$$q_{d_i}(H) = t\pi_{d_i} r + z_{d_i} p^\perp \quad \text{for } i = 1, 2,$$
$$H((d_1 d_2)_1) = t\pi_{d_1}\pi_{d_2} r \cdot p,$$
$$H((d_1)_i \oplus (d_2)_j) = H((d_1)_i)H((d_2)_j) \quad \text{for } i, j = 0, 1,$$
$$H(M) = 0 \quad \text{if } E(M) \not\subseteq \{d_1, d_2\}.$$

Then H is a *collinear pairlike* function. If $\pi_{d_1}, \pi_{d_2} \neq 0$, it is *collinear strictly pairlike*. It is obviously a strong Tutte function.

An equivalent definition begins with p, $\boldsymbol{\pi}$, and p_e as above. Let $q_{d_1}, q_{d_2} \in K^2$ be arbitrary vectors satisfying $p \cdot (\pi_{d_1} q_{d_2} - \pi_{d_2} q_{d_1}) = 0$. Define a function H on $\mathscr{M}(U)$ as before except that

$$q_{d_i}(H) = q_{d_i} \quad \text{for } i = 1, 2,$$
$$H((d_1 d_2)_1) = \pi_{d_1} p \cdot q_{d_2} = \pi_{d_2} p \cdot q_{d_1}.$$

When one or both p_{d_i} are zero, this example falls under Example 2.5B or A.

Example 2.5 (Multiplicative discrete functions). Let $\mathbf{x}, \mathbf{y} \in K^U$ and define F on $\mathscr{M}(U)$ by

$$F(M) = \begin{cases} \prod_{e \in E_1(M)} x_e \cdot \prod_{e \in E_0(M)} y_e & \text{if } M \text{ is discrete}, \\ 0 & \text{if } M \text{ is not discrete}. \end{cases}$$

We call F a *discrete* function because it is zero on nondiscrete matroids and *multiplicative* because it obeys (M). A nil function is one example.

Example 2.5A (Paranil functions). A multiplicative discrete function on $\mathscr{M}(U)$ is obviously a strong Tutte function with parameters $\mathbf{a} = \mathbf{b} = \mathbf{0}$. We call it *paranil* because it has zero parameters.

Example 2.5B (Pointlike functions). Here there is one special point $d \in U$. Choose a nonzero $p = (a, b) \in K^2$ and any $\boldsymbol{\pi} \in K^{U\setminus\{d\}}$. Set $p_e = \pi_e p$ if $e \neq d$ and choose any $p_d \in K^2$. Let $p^\perp = (b, -a)$. For arbitrary $w \in K$, set

$$q_d = wp^\perp, \quad \text{but } q_e = 0 \text{ if } e \neq d. \tag{2.3}$$

This determines a multiplicative discrete function on $\mathscr{M}(U)$, which is easily seen to be a strong Tutte function. We call it a *pointlike* function. It is *strictly pointlike* if $\boldsymbol{\pi} \neq \mathbf{0}$ or $p_d \neq 0$; otherwise it is paranil.

One can easily verify that a nonnil function whose domain contains all digons is pointlike if and only if it is a strong Tutte function whose point-value vector is zero at all but one point.

Now the first main theorem.

Theorem 2.1. *The strong Tutte functions with point universe U and with any minor-closed domain $\mathscr{M} \supseteq \mathscr{M}_{(3)}(U)$, in particular the global strong Tutte functions over U, are precisely the normal, nil, elementary, planar and collinear pairlike, paranil, and pointlike functions.*

A nice way to state the classification of global strong Tutte functions is that they are determined by precisely the parameter-point-value quadruples $(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y})$ in the following list:

$(\mathbf{0}, \mathbf{0}; \mathbf{x}, \mathbf{y})$	(paranil),
$(\mathbf{a}, \mathbf{0}; \mathbf{a}u, \mathbf{y})$	(primal elementary),
$(\mathbf{0}, \mathbf{b}; \mathbf{x}, \mathbf{b}v)$	(dual elementary),
$(\mathbf{a}, \mathbf{b}; \mathbf{0}, \mathbf{0})$	(nil; $F(\varnothing) = 0$ or 1),
$(\mathbf{a}, \mathbf{b}; \mathbf{a}u + \mathbf{b}, \mathbf{a} + \mathbf{b}v)$	(normal),
$((a_1, a_2, 0, \ldots), (b_1, b_2, 0, \ldots);$ $(ua_1 + wb_1, ua_2 + wb_2, 0, \ldots),$ $(wa_1 + vb_1, wa_2 + vb_2, 0, \ldots))$	(planar pairlike),
$((a, \pi a, 0, \ldots), (b, \pi b, 0, \ldots);$ $(tr_1 + z_1 b, t\pi r_1 + z_2 b, 0, \ldots),$ $(tr_0 - z_1 a, t\pi r_0 - z_2 a, 0, \ldots))$	(collinear pairlike),
$((a_1, a\boldsymbol{\pi}), (b_1, b\boldsymbol{\pi}); (wb, \mathbf{0}), (-wa, \mathbf{0}))$	(pointlike).

Important domains to which Theorem 2.1 applies are the classes of matroids of outerplanar graphs or series-parallel networks, graphic matroids, regular matroids, transversal matroids, the matroids representable over a fixed field, or the matroids having no minor in any fixed list of matroids on four or more points.

The second main theorem concerns Tutte functions of colored matroids. A *colored matroid* is a pair (M, κ) consisting of a matroid M and a coloring κ, which is a mapping $E(M) \to C$, where C is a fixed *color set*. A *minor* of (M, κ) is a colored matroid $(M', \kappa|_{E(M')})$, where M' is a minor of M.

(We may write (M', κ), understanding the restriction of κ to $E(M')$.) An *isomorphism* of colored matroids is a matroid isomorphism which preserves colors. If $\mathscr{M}$ is a class of matroids and C is a color set, the class of *C-colored matroids in* $\mathscr{M}$ is

$$[\mathscr{M}, C] = \{(M, \kappa) : M \in \mathscr{M} \text{ and } \kappa : E(M) \to C\}.$$

An *invariant* of $[\mathscr{M}, C]$ is a function on $[\mathscr{M}, C]$ for which $F(M_1, \kappa_1) = F(M_2, \kappa_2)$ whenever (M_1, κ_1) and (M_2, κ_2) are isomorphic as colored matroids. A *strong Tutte function* (or, *invariant*) *of C-colored matroids in* $\mathscr{M}$ is a function (or an invariant) F defined on $[\mathscr{M}, C]$, where $\mathscr{M}$ is a point-covering minor-closed class in U, together with parameters a_c and b_c for all $c \in C$, satisfying

$$(\mathrm{M_C}) \qquad F(M, \kappa) = F(M_1, \kappa)F(M_2, \kappa)$$

if $M = M_1 \oplus M_2$ and

$$(\mathrm{L_C}) \qquad F(M, \kappa) = a_{\kappa(e)}F(M \backslash e, \kappa) + b_{\kappa(e)}F(M/e, \kappa)$$

if e is a nonseparator of M_1. A strong Tutte function of colored matroids is *global* if its domain is $[\mathscr{M}(U), C]$. It has *global type* if it is a restriction of a global strong Tutte function; loosely speaking, we may then call it *global* if the domain is made plain by context.

To define normal, nil, elementary, and paranil functions of colored matroids, simply replace the parameters $p_e = (a_e, b_e)$ by $p_{\kappa(e)} = (a_{\kappa(e)}, b_{\kappa(e)})$ in Examples 2.1–2.3 and 2.5A, the matroids $(e)_r$ by $((e)_r, c)$, and q_e by $q_{(e,c)}$ for $e \in U, c \in C$. The (parametrized) *rank polynomial* of a colored matroid (M, κ), with parameter sequences $\mathbf{a}, \mathbf{b} \in K^C$, is

$$R_{(M,\kappa)}(\mathbf{a}, \mathbf{b}; u, v) = \sum_{S \subseteq E} u^{\operatorname{cork} S} v^{\operatorname{nul} S} \cdot \prod_{e \in S^c} a_{\kappa(e)} \cdot \prod_{e \in S} b_{\kappa(e)}.$$

A normal function of colored matroids is just an evaluation of this polynomial.

Theorem 2.2. *Let $\mathscr{M}$ be a point-complete, minor-closed class of matroids in U containing a triangle and a triad and let C be a nonvoid set. The strong Tutte functions of C-colored matroids in $\mathscr{M}$ are the normal, nil, elementary, and paranil functions of colored matroids. The strong Tutte invariants of C-colored matroids in $\mathscr{M}$ are the normal and nil functions as well as those elementary and paranil ones for which* $\mathbf{x}$ *and* $\mathbf{y}$ *are functions of color alone.*

Of the seven global types of strong Tutte function the normal type is the most important, both because it is the exact generalization of Tutte-Grothendieck invariants—since it is given by a rank polynomial—and because it is the only nondegenerate type and (besides nil functions, which are trivial) the only type which exists for all parameters, at least when the universe has more than two points. So it is worth knowing when a function of another global type really is abnormal. In reading the following proposition, keep in mind the following fact presented below (Theorem 6.1): a global strong Tutte function is necessarily paranil if $\mathbf{a} = \mathbf{b} = \mathbf{0}$, primal elementary if $\mathbf{a} \neq \mathbf{0} = \mathbf{b}$, and dual elementary if $\mathbf{a} = \mathbf{0} = \mathbf{b}$.

Proposition 2.3. *A strong Tutte function F is normal if and only if*

(1) *it is nil, and $F(\varnothing) = 1$, if F is paranil,*
(2) $\mathbf{y} = \mathbf{a}$, *if F is primal elementary,*
(3) $\mathbf{x} = \mathbf{b}$, *if F is dual elementary.*

When neither $\mathbf{a}$ nor $\mathbf{b}$ is $\mathbf{0}$, F is normal if and only if

(4) *the parameter vectors are collinear, if F is nil,*
(5) $w = 1$ *or* $p_{d_1} \| p_{d_2}$, *if F is planar pairlike,*
(6) *there is $q \in K^2$ such that $q_e = \pi_e q$ for all e, if F is collinear pairlike,*
(7) $p_e = 0$ *for all $e \neq d$, or $a_e x_d = -|p_d, p_e|$ (equivalently, $b_e y_d = |p_d, p_e|$) for some (equivalently, all) $e \neq d$ such that $p_e \neq 0$, if F is pointlike,*
(8) $q_d = 0$, *if F is pointlike and $p_e \| p_d$ for some $e \neq d$ such that $p_e \neq 0$.*

We omit the proof. One can be based on §§4 and 5.

3. Reduction and expansion of domain

The crucial properties of a strong Tutte function live on small matroids, having up to three points. Given a minor-closed class $\mathcal{M}$, obviously a strong Tutte function on $\mathcal{M}$ restricts to a strong Tutte function on $\mathcal{M}_{(3)}$. Conversely,

Theorem 3.1. *Any strong Tutte function on $\mathcal{M}_{(3)}$ extends to a unique strong Tutte function on $\mathcal{M}$.*

Proof. Let F be a strong Tutte function on $\mathcal{M}_{(3)}$. Extend it inductively to a function on $\mathcal{M}$ by taking, for each $M \in \mathcal{M} \backslash \mathcal{M}_{(3)}$, either a direct decomposition $M = M_1 \oplus M_2$ (if possible) and defining

$$F(M) = F(M_1)F(M_2), \tag{3.1}$$

or else a nonseparating $e \in M$ and setting

$$F(M) = a_e F(M \backslash e) + b_e F(M \backslash e). \tag{3.2}$$

This is certainly the only way to extend F to a strong Tutte function on $\mathcal{M}$. The question is whether it is consistent.

Supposing it were not, there would be a smallest $M \in \mathcal{M}$ for which the extended F is inconsistently defined. That is, either M has a direct-sum representation $M = M_1' \oplus M_2'$ with

$$F(M) \neq F(M_1')F(M_2'), \tag{3.1$'$}$$

or it has a nonseparating element f such that

$$F(M) \neq a_f F(M \backslash f) + b_f F(M / f). \tag{3.2$'$}$$

Suppose (3.1) applies to M. If inequality (3.1′) also holds, let $M_{ij} = M|[E(M_i) \cap E(M_j')]$, so that for instance $M_i = M_{i1} \oplus M_{i2}$. Since M is a smallest counterexample, $F(M_1)F(M_2) = \prod F(M_{ij}) = F(M_1')F(M_2')$, contradicting (3.1′).

On the other hand if (3.2′) holds for M, say with $f \in M_1$, then calculating $F(M_1)$ yields $F(M_1)F(M_2) = a_f F(M \backslash f) + b_f F(M/f)$, contradicting (3.2′). If M satisfies (3.2) and (3.1′), a similar calculation yields a contradiction.

This discussion shows that M is connected. Since $M \notin \mathcal{M}_{(3)}$, it has at least four points.

Lemma 3.2. *Let e, f be nonseparating points in a matroid N. The following statements are equivalent:*

(i) *e is a separator in $N\backslash f$.*
(ii) *e is a coloop in $N\backslash f$.*
(iii) *e and f are in series in N.*
(iv) *f is a separator in $N\backslash e$.* □

Now define $D = \{e \in M : (3.2) \text{ holds for } e\}$. Let $e \in D$ and $f \in D^c$. Suppose that f is a separator in neither $M\backslash e$ nor M/e. Therefore

$$\begin{aligned} F(M) &= a_e F(M\backslash e) + b_e F(M/e) \\ &= a_e a_f F(M\backslash ef) + a_e b_f F(M\backslash e/f) + b_e a_f F(M/e\backslash f) + b_e b_f F(M/ef). \end{aligned}$$

By Lemma 3.2 and its dual, e is a nonseparator in $M\backslash f$ and M/f. As a consequence, the last expression

$$= a_f F(M\backslash f) + b_f F(M/f).$$

But this contradicts the hypothesis $f \in D^c$. We conclude that f is a separator in $M\backslash e$ or M/e, hence e and f are in series or in parallel in M whenever $e \in D$ and $f \in D^c$.

When $\{e, f\}$ and $\{e', f'\}$, where $e, e' \in D$ and $f, f' \in D^c$, have one element in common, it is impossible for one to be a circuit and the other a cocircuit. We deduce that either $\{e, f\}$ is always a circuit (for $e \in D$ and $f \in D^c$), in which case M is a cocircuit, or $\{e, f\}$ is always a cocircuit, in which case M is a circuit.

Say M is a circuit. (The opposite case is treated dually.) Let $E(M) = \{e_1, e_2, \ldots, e_n\}$ with $e_1 \in D$ and $e_2 \in D^c$. Thus

$$\begin{aligned} F(M) &= a_{e_1} F(M\backslash e_1) + b_{e_1} F(M/e_1) \\ &= a_{e_1} x_{e_2} x_{e_3} \cdots x_{e_n} + b_{e_1} a_{e_2} x_{e_3} \cdots x_{e_n} + b_{e_1} b_{e_2} F(M/e_1 e_2) \\ &= (a_{e_1} x_{e_2} + b_{e_1} a_{e_2}) x_{e_3} \cdots x_{e_n} + b_{e_1} b_{e_2} F(M/e_1 e_2), \end{aligned}$$

and

$$\begin{aligned} F(M) &\neq a_{e_2} F(M\backslash e_2) + b_{e_2} F(M/e_2) \\ &= (a_{e_2} x_{e_1} + b_{e_2} a_{e_1}) x_{e_3} \cdots x_{e_n} + b_{e_2} b_{e_1} F(M/e_2 e_1). \end{aligned}$$

Combining and simplifying,

$$0 \neq \begin{vmatrix} x_{e_1} - b_{e_1} & x_{e_2} - b_{e_2} \\ a_{e_1} & a_{e_2} \end{vmatrix} x_{e_3} \ldots x_{e_n}. \tag{3.3}$$

On the other hand, performing the same calculation in $(e_1 e_2 e_3)_2 \in \mathscr{M}_{(3)}$ shows that (3.3) should be an equality. This is a contradiction.

It follows that no M can exist and Theorem 3.1 is proved. □

Given a class $\mathscr{M}$ of matroids in U, not necessarily minor-closed, let

$$\overline{\mathscr{M}} = \{M \in \mathscr{M}(U) : \mathscr{M}_{(3)}(M) \subseteq \mathscr{M}\}$$

be the class of matroids in U whose digon, triangle, and triad minors lie in $\mathscr{M}$. If $\mathscr{M}_{(3)} \subseteq \mathscr{M}$, then $\overline{\mathscr{M}} \supseteq \mathscr{M}$. The theorem implies that

$$\mathscr{M}[\mathbf{p}; \mathbf{q}] = \overline{\mathscr{M}_{(3)}[\mathbf{p}; \mathbf{q}]}. \tag{3.4}$$

Let F be a function defined on $\mathscr{M}$. We say F *satisfies the multiplicative decomposition* $M = M_1 \oplus M_2$ if $M, M_1, M_2 \in \mathscr{M}$ and (3.1) holds. We say F *satisfies the linear decomposition* (M, e) if e is a nonseparating point of M and (3.2) holds.

Corollary 3.3. *Let F be a function defined on a class $\mathscr{M}$ of matroids such that $\mathscr{M} \supseteq \mathscr{M}_{(3)}$. Suppose F is a strong Tutte function on $\mathscr{M}_{(3)}$ and satisfies a decomposition of each $M \in \mathscr{M} \backslash \mathscr{M}_{(3)}$. Then F extends uniquely to a strong Tutte function on $\overline{\mathscr{M}}$.*

Proof. $F|_{\mathscr{M}_{(3)}}$ extends uniquely to a strong Tutte function $\overline{F}$ on $\overline{\mathscr{M}}$, which necessarily agrees with F on $\mathscr{M}$. □

A weakening of (M) is the following property of *discrete multiplicativity*:

(DM) $$F(M) = \prod_{e \in E_1(M)} x_e \cdot \prod_{e \in E_0(M)} y_e \quad \text{if } M \text{ is discrete.}$$

As one would expect, this can substitute for (M). Corollary 3.3 is one means of proving this fact.

Corollary 3.4. *A function defined on a minor-closed class of matroids, which satisfies* (L) *and* (DM), *also satisfies* (M) *and is a strong Tutte function.*

Proof. Let F be the function and $\mathscr{M}$ its domain. F is a strong Tutte function on $\mathscr{M}_{(3)}$ because every matroid in $\mathscr{M}_{(3)}$ is discrete or connected. F satisfies a decomposition of every $M \in \mathscr{M} \backslash \mathscr{M}_{(3)}$: multiplicative if M is discrete, linear otherwise. Apply Corollary 3.3. □

4. The effect of digons

A function F defined on a minor-closed class $\mathscr{M}$ of matroids in U is sharply constrained already by being a Tutte function on two-point circuits. Suppose $(ef)_1 \in \mathscr{M}$ and F is a Tutte function (weak or strong, which are equivalent here) on $\mathscr{M}((ef)_1)$. By deleting and contracting either e or f we get two equal expressions for $F((ef)_1)$:

(4.1) $$F((ef)_1) = a_e x_f + b_e y_f = a_f x_e + b_f y_e.$$

Thus we have

Lemma 4.1. *Let F be defined on $\mathscr{M}((ef)_1)$. F is a weak Tutte function on $\mathscr{M}((ef)_1)$ if and only if $p_e \cdot q_f = p_f \cdot q_e = F((ef)_1)$.* □

If p_e and p_f are linearly independent, there is a unique matrix A_{ef} such that $(q_e, q_f) = A_{ef}(p_e, p_f)$. If also F is a weak Tutte function on $\mathscr{M}((ef)_1)$, Lemma 4.1 implies that A_{ef} is symmetric.

Suppose F is a weak Tutte function on $\mathscr{M}_{(2)}$ and $(e_1e_2)_1, (e_1e_3)_1, (e_2e_3)_1 \in \mathscr{M}$. Then (4.1) gives three equations:

$$\begin{aligned} b_2 y_1 - b_1 y_2 \qquad\qquad &= -a_2 x_1 + a_1 x_2, \\ b_3 y_2 - b_2 y_3 &= \qquad\quad -a_3 x_2 + a_2 x_3, \\ -b_3 y_1 \qquad + b_1 y_3 &= a_3 x_1 \qquad\qquad - a_1 x_3. \end{aligned}$$

Multiplying respectively by b_3, b_1, b_2, adding up the equations, and rearranging, we get the first equation in (4.2); the second is proved similarly.

$$\text{(4.2)}\qquad \begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} y_1 & y_2 & y_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

Suppose p_1, p_2 are linearly independent. Then $(q_1, q_2, q_3) = A(p_1, p_2, p_3)$ for some matrix A. But A is determined by e_1 and e_2 and is symmetric. Varying e_3, we obtain

Lemma 4.2. *Let F be a weak Tutte function on $\mathscr{M}_{(2)}$. Let U have elements e_1 and e_2 for which p_{e_1} and p_{e_2} span K^2 and suppose $(e_1e_2)_1, (e_1e)_1, (e_2e)_1 \in \mathscr{M}$ for all $e \in U\backslash\{e_1, e_2\}$. Then there exist $\alpha, \beta, \gamma \in K$ such that*

$$\text{(4.3)}\qquad q_e = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} p_e \quad \textit{for all } e \in U,$$

and there is no other 2-by-2 matrix A such that $q_e = Ap_e$ for all $e \in U$.

Suppose on the other hand that F is a weak Tutte function on $\mathscr{M}_{(2)}$ and in U the vectors p_e are collinear but not all zero; that is, there is a nonzero vector $p = (a, b) \in K^2$ so that $p_e = \pi_e p$ for all $e \in U$. Let $U^* = \{e \in U : p_e \neq 0\}$. Lemma 4.1 takes the following form:

$$\text{(4.4)}\qquad p \perp (q_e/\pi_e - q_f/\pi_f) \quad \text{if } e, f \in U^* \text{ and } (ef)_1 \in \mathscr{M};$$

while $\pi_e p \cdot q_f = p_f \cdot q_e = 0$ if $e \in U^*$, $f \in U\backslash U^*$, and $(ef)_1 \in \mathscr{M}$, in other words

$$\text{(4.5)}\qquad p \perp q_f \quad \text{if } f \in U\backslash U^* \text{ and } (\exists e \in U^*)(ef)_1 \in \mathscr{M}.$$

Let $G_2(W)$ be the graph whose vertex set is $W \subseteq U$ and whose edges are the pairs $\{e, f\}$ whose digon $(ef)_1 \in \mathscr{M}$. If $G_2(U^*)$ is connected, then (4.4) holds for all $e, f \in U^*$; so there is a vector $r \in K^2$ such that $q_e = \pi_e r + \varepsilon_e p^\perp$ for all $e \in U^*$, where $p^\perp = (b, -a)$ is orthogonal to p. If in addition for every $f \in U\backslash U^*$ there is an $e \in U^*$ which is adjacent to f in $G_2(U)$, then $q_f = \varepsilon_f p^\perp$ for each $f \in U\backslash U^*$.

Lemma 4.3. *Let F be a weak Tutte function on $\mathscr{M}_{(2)}$. Suppose the parameter vectors p_e, $e \in U$, are collinear but not all zero. Let $U^* = \{e \in U : p_e \neq 0\}$. Suppose that $G_2(U^*)$ is connected and every point of $U\backslash U^*$ is adjacent in $G_2(U)$ to a point of U^*. Choose $p = (a, b) \neq 0$ to span the vectors p_e, so $p_e = \pi_e p$ for $e \in U$, and let $p^\perp = (b, -a)$. Then there exist $r \in K^2$ and $\boldsymbol{\varepsilon} \in K^U$ so that $q_e = \pi_e r + \varepsilon_e p^\perp$ for all $e \in U$.* □

Let us define two types of function on $\mathscr{M}_2(U)$.

Example 4.1 (Planar functions). Suppose U and $\mathbf{a}, \mathbf{b} \in K^U$ are given. Let $u, v, w \in K$ be arbitrary. Define a function F on $\mathscr{M}_{(2)}(U)$ by

$$F(\varnothing) = 1,$$

$$q_e = \begin{pmatrix} u & w \\ w & v \end{pmatrix} p_e,$$

$$F((ef)_1) = p_e \cdot q_f = p_f \cdot q_e = (a_e a_f)u + (a_e b_f + b_e a_f)w + (b_e b_f)v.$$

We call F a *planar* function on $\mathscr{M}_{(2)}(U)$ since in general the parameter vectors, and also the point-value vectors, will span K^2. If the parameter vectors do span K^2, we call the function *truly planar.* A planar function is obviously a strong Tutte function on $\mathscr{M}_{(2)}(U)$.

Example 4.2 (Collinear functions). Suppose U, $\boldsymbol{\pi} \in K^U$, and nonorthogonal vectors $p = (a, b)$ and $r = (r_1, r_0) \in K^2$ are given. Set $p_e = \pi_e p$ (so $\mathbf{a} = \boldsymbol{\pi} a$ and $\mathbf{b} = \boldsymbol{\pi} b$) and $p^\perp = (b, -a)$. Let $t \in K$ and $\mathbf{z} \in K^U$ be arbitrary. Define $F : \mathscr{M}_{(2)}(U) \to K$ by

$$F(\varnothing) = 1,$$
$$q_e = t\pi_e r + z_e p^\perp,$$
$$F((ef)_1) = p_e \cdot q_f = p_f \cdot q_e = t\pi_e \pi_f p \cdot r.$$

We call F a *collinear* function on $\mathscr{M}_{(2)}(U)$ since the parameter vectors are homogeneously collinear and the normalized point-value vectors q_e/π_e (for $\pi_e \neq 0$) lie on an affine line. A collinear function is obviously a strong Tutte function on $\mathscr{M}_{(2)}(U)$.

A notation we use in connection with collinear functions is the vector $\varphi_e = (\pi_e, z_e)$.

Proposition 4.4. *A planar function is a collinear function precisely when the parameter vectors p_e are collinear. A collinear function is planar if and only if the point-value vectors are collinear.* □

Proposition 4.5. *Let U and parameters $\mathbf{a}, \mathbf{b} \in K^U$ be given and set $p_e = (a_e, b_e)$.*

(a) *If the parameter vectors p_e, $e \in U$, span K^2, then the weak Tutte functions on $\mathscr{M}_{(2)}(U)$ with parameters $\mathbf{a}, \mathbf{b}$ are precisely the planar functions. Two such functions are equal if and only if $(u, v, w) = (u', v', w')$.*

(b) *If the vectors p_e, $e \in U$, are collinear but not all zero, say $p_e = \pi_e p$ where $p = (a, b) \in K^2$, then the weak Tutte functions on $\mathscr{M}_{(2)}(U)$ with parameters $\mathbf{a}, \mathbf{b}$ are precisely the collinear functions. Two such functions are equal if and only if*

$$\mathbf{z} - \mathbf{z}' = \left| t'r' - tr, \quad \frac{p}{|p \cdot p|} \right| \boldsymbol{\pi}.$$

Proof. (a) Apply Lemma 4.2.

(b) Lemma 4.3 implies the function is collinear. Suppose two collinear functions are equal. That means

$$\pi_e(t'r' - tr) = (z_e - z'_e)p^\perp \quad \text{for all } e \in U.$$

If $\pi_e \neq 0$, $t'r' - tr = \pi_e^{-1}(z_e - z'_e)p^\perp$. Since this quantity is independent of $e \in U^*$, there must be a scalar constant α such that $z_e - z'_e = \pi_e \alpha$ for all $e \in U^*$ and $t'r' = tr + \alpha p^\perp$. Thus $\alpha p \cdot p = |t'r' - tr,\ p|$. If $\pi_e = 0$, then $z_e - z'_e = 0 = \pi_e \alpha$. This settles the conditions under which the two functions are equal. □

5. The effect of triangles and triads

We now consider a function F on a minor-closed class $\mathscr{M}$ which is a Tutte function on $\mathscr{M}_{(2)}$ with respect to a parameter sequence $\mathbf{p}$. When is it a strong

Tutte function on $\mathscr{M}_{(3)}$? The answer depends in part on how big $\mathscr{M}_{(3)}$ is, but since even one triangle or triad is very influential we begin by examining their individual effects. We define $\mathscr{M}_{(3)2} = \{M \in \mathscr{M}_{(3)} : |E(M)| \le 2\}$.

Lemma 5.1. *Let F be a function on a minor-closed class $\mathscr{M}$ of matroids which is a strong Tutte function on $\mathscr{M}_{(3)2}$ with parameters $\mathbf{a}, \mathbf{b} \in K^U$.*

(a) *Suppose $C = (e_1e_2e_3)_2 \in \mathscr{M}$. Then F is a strong Tutte function on $\mathscr{M}(C)$ if and only if it satisfies a decomposition of C and*

$$x_k \begin{vmatrix} x_i - b_i & x_j - b_j \\ a_i & a_j \end{vmatrix} = 0 \tag{5.1}$$

for all permutations ijk of $\{1, 2, 3\}$.

(b) *Suppose $D = (e_1e_2e_3)_1 \in \mathscr{M}$. Then F is a strong Tutte function on $\mathscr{M}(D)$ if and only if F satisfies a decomposition of D and*

$$y_k \begin{vmatrix} y_i - a_i & y_j - a_j \\ b_i & b_j \end{vmatrix} = 0 \tag{5.2}$$

for all permutations ijk of $\{1, 2, 3\}$.

Proof. We prove (a); the proof of (b) is dual.

Suppose F is a strong Tutte function on $\mathscr{M}(C)$. Deleting and contracting first in the order e_ie_j and then in the opposite order, we obtain

$$\begin{aligned} F(C) &= a_ix_jx_k + b_ia_jx_k + b_ib_jy_k \\ &= a_jx_ix_k + b_ja_ix_k + b_jb_iy_k, \end{aligned} \tag{5.3}$$

which simplifies to (5.1).

Suppose on the other hand (5.1) holds for all $k = 1, 2, 3$. We wish to prove that the value of

$$\alpha_{ijk} = a_ix_jx_k + b_ia_jx_k + b_ib_jy_k$$

is independent of the ordering ijk. Equation (5.1) expresses the fact that $\alpha_{ijk} = \alpha_{jik}$. That F is a Tutte function on $\mathscr{M}_{(2)}$ implies $\alpha_{ijk} = \alpha_{ikj}$. It follows that all α_{ijk} are equal. Since $F(C)$ equals one of them by hypothesis, F is a strong Tutte function on $\mathscr{M}(C)$. □

When F is a planar function (5.1) simplifies to

$$(w-1)|p_i, p_j| p_k \cdot (u, w) = 0 \tag{5.4}$$

and (5.2) to

$$(w-1)|p_i, p_j| p_k \cdot (w, v) = 0. \tag{5.5}$$

Lemma 5.2. *Let F be a function, defined on a minor-closed class $\mathscr{M}$, which is planar on $\mathscr{M}_{(2)}$ with respect to parameters $\mathbf{a}, \mathbf{b}$.*

(a) *If F is a strong Tutte function on $\mathscr{M}(C)$ for some $C = (efg)_2 \in \mathscr{M}$, then either F is normal, or $u = w = 0$ (so $\mathbf{x} = \mathbf{0}$), or p_e, p_f, p_g are collinear, or exactly two of p_e, p_f, p_g are orthogonal to (u, w) (so exactly the corresponding two of x_e, x_f, x_g are zero).*

(b) *Under a similar hypothesis on $\mathscr{M}(D)$ for $D = (efg)_1 \in \mathscr{M}$, either F is normal, or $w = v = 0$ (so $\mathbf{y} = \mathbf{0}$), or p_e, p_f, p_g are collinear, or exactly two of p_e, p_f, p_g are orthogonal to (w, v) (so exactly two of y_e, y_f, y_g are zero).* □

When F is a collinear function (5.1) simplifies to

(5.6) $$ab|\varphi_i, \varphi_j|\varphi_k \cdot (tr_1, b) = 0$$

and (5.2) to

(5.7) $$ab|\varphi_i, \varphi_j|\varphi_k \cdot (tr_0, -a) = 0,$$

where φ_e denotes (π_e, z_e). Thus:

Lemma 5.3. *Let F be a function, defined on a minor-closed class $\mathscr{M}$, which is collinear on $\mathscr{M}_{(2)}$ with respect to parameters $\mathbf{p} = \boldsymbol{\pi} p$, where $p = (a, b)$.*

(a) *If F is a strong Tutte function on $\mathscr{M}(C)$ for some $C = (efg)_2 \in \mathscr{M}$, then either a or $b = 0$, or $\varphi_e, \varphi_f, \varphi_g$ are all parallel, or exactly two of $\varphi_e, \varphi_f, \varphi_g$ are parallel to $(-b, tr_1)$ (so the two corresponding $x_h = 0$).*

(b) *If F is a strong Tutte function on $\mathscr{M}(D)$ for some $D = (efg)_1 \in \mathscr{M}$, then either a or $b = 0$, or $\varphi_e, \varphi_f, \varphi_g$ are all parallel, or exactly two of $\varphi_e, \varphi_f, \varphi_g$ are parallel to (a, tr_0) (so the two corresponding $y_h = 0$).* □

6. Provings and improvings of the main theorems

We are ready to prove a slight strengthening of Theorem 2.1.

Theorem 6.1. *Let $\mathscr{M}$ be a minor-closed class of matroids in a set U, let $\mathbf{a}, \mathbf{b} \in K^U$ be fixed parameters, and assume $\mathscr{M}$ contains every two-point circuit in U. The strong Tutte functions on $\mathscr{M}$ with parameters $\mathbf{a}$ and $\mathbf{b}$ are precisely*

(1) *the multiplicative discrete functions, if $\mathbf{a} = \mathbf{b} = \mathbf{0}$;*
(2) *the primal elementary functions, if $\mathbf{b} = \mathbf{0}$ but $\mathbf{a} \neq \mathbf{0}$;*
(3) *the dual elementary functions, if $\mathbf{a} = \mathbf{0}$ but $\mathbf{b} \neq \mathbf{0}$;*
(4) *the normal, pointlike, and nil functions and the planar and collinear pairlike functions, if $\mathbf{a}$ and $\mathbf{b} \neq \mathbf{0}$ and $\mathscr{M}$ contains every three-point circuit and cocircuit.*

Proof. The case $\mathbf{a} = \mathbf{b} = \mathbf{0}$ is obvious.

If $\mathbf{b} = \mathbf{0} \neq \mathbf{a}$, a strong Tutte function F is, by Proposition 4.4(b), a collinear function for which $b = 0 \neq a$. Then $x_e = \pi_e x$, where $x = tr_1$, and $y_e = t\pi_e r_0 - z_e a$ is completely arbitrary. Thus F is primal elementary.

The case $\mathbf{a} = \mathbf{0} \neq \mathbf{b}$ is similar.

Henceforth F is a strong Tutte function on $\mathscr{M}$ with parameters $\mathbf{a}, \mathbf{b} \neq 0$ and with $\mathscr{M} \supseteq \mathscr{M}_{(3)}$. The proof splits according as the parameter vectors p_e span K^2 or not.

If they span K^2, Proposition 4.5(a) applies. Assume F is abnormal, so $w \neq 1$. Choose d_1 and d_2 so p_{d_1} and p_{d_2} span K^2. Letting $e = d_1$ and $f = d_2$ in Lemma 5.2 yields $x_g = y_g = 0$ for all $g \neq d_1, d_2$.

If $p_g = 0$ for all $g \neq d_1, d_2$, then F is planar pairlike. Otherwise there is a $p_{d_3} \neq 0$ such that, say, p_{d_1} and p_{d_3} are linearly independent. From Lemma 5.2 with $e = d_1$, $f = d_3$, and $g = d_2$, we deduce $q_{d_2} = 0$. If any p_g is independent of p_{d_2} we can similarly deduce $q_{d_1} = 0$, whence F is nil. The alternative is that $p_e = \pi_e p$ for $e \neq d_1$, where $p = (a, b) \neq 0$. From Lemma 4.1 we have $q_{d_1} \cdot p = 0$; it follows that q_{d_1} is a scalar multiple of $p^\perp = (b, -a)$ and therefore F is pointlike.

This concludes the case in which the parameter vectors span K^2. Notice that we used only the existence in $\mathscr{M}$ of 3-point circuits and cocircuits on all triples $\{d_1, d_2, g\}$, where d_1 and d_2 are a fixed pair whose parameter vectors are independent.

From now on we assume each $p_e = \pi_e p$ for some vector $p = (a, b)$ in which neither component is zero, and not all π_e are zero. Proposition 4.5(b) shows that F is collinear.

If U contains points d_1 and d_2 whose vectors φ_{d_1} and φ_{d_2} are linearly independent, then Lemma 5.3 with $e = d_1$ and $f = d_2$ implies

$$z_g = -\frac{tr_1}{b} = \frac{tr_0}{a} \quad \text{for all } g \neq d_1, d_2,$$

whence $t(p \cdot r) = 0$. It follows that $t = 0$, so $q_g = z_g p^{\perp}$ for all $g \in U$ and $q_g = 0$ for $g \neq d_1, d_2$. If $\pi_e \neq 0$ for some $e \neq d_1, d_2$, then φ_e is linearly independent of φ_{d_1} or φ_{d_2}, let us say φ_{d_1}, and from Lemma 5.3 with $f = d_1$ and $g = d_2$ we deduce $z_{d_2} = 0$, whence $q_{d_2} = 0$. Therefore F is pointlike. If on the contrary $\pi_e = 0$ for all $e \neq d_1, d_2$, then F is a collinear pairlike function because $\pi_{d_1} q_{d_2} - \pi_{d_2} q_{d_1}$ is a scalar multiple of $p^{\perp}$.

Suppose finally that the vectors φ_e, $e \in U$, are collinear. Because $\boldsymbol{\pi} \neq \mathbf{0}$, $\mathbf{z} = \theta \boldsymbol{\pi}$ for some $\theta \in K$ and we have $q_e = \pi_e(tr + \theta p^{\perp})$. Thus F is normal. This completes the classification. □

We shall deduce Theorem 2.2 from a generalization of Theorems 2.1 and 6.1 to colored universal point sets. A *colored point universe* (U, γ) is a universe U together with a mapping $\gamma : U \to C$, called a *coloring* of U, whose codomain is a *color set* C. A *color class* is any nonempty set $\gamma^{-1}(c)$ for $c \in C$. The coloring itself is usually less important than the partition into color classes and the corresponding equivalence relation on U, which we denote by $\equiv$. We may even call $(U, \equiv)$ a colored universe, ignoring the coloring itself.

A *colored isomorphism* of matroids M_1 and M_2 in a colored universe is an isomorphism $\theta : E(M_1) \to E(M_2)$ which preserves color class; that is, $e \equiv \theta(e)$ for all $e \in E(M_1)$. By $M_1 \equiv M_2$ we mean M_1 and M_2 are color-isomorphic. A function φ on U is *equivariant* if it is constant on color classes. A function F of matroids in U is called *equivariant* when it satisfies

(E) $$M_1 \equiv M_2 \Rightarrow F(M_1) = F(M_2).$$

A strong or weak *Tutte equivariant* of matroids in $(U, \equiv)$ is a Tutte function of matroids in U which is an equivariant function and whose parameter sequence is also equivariant. For a strong Tutte function, equivariance is implied by

(E_1) $$e \equiv f \Rightarrow p_e = p_f \text{ and } q_e = q_f.$$

Here is the main theorem.

Theorem 6.2. *Assume a colored point universe $(U', \equiv)$, equivariant parameters $\mathbf{a}, \mathbf{b} \in K^{U'}$, and a point-covering minor-closed class $\mathscr{M}'$ of matroids in U' are given such that, for any $\{e_1, e_2\} \subseteq U'$, there is a circuit $(f_1 f_2)_1 \in \mathscr{M}'$ with $e_1 \equiv f_1$ and $e_2 \equiv f_2$. Then the strong Tutte equivariants defined on $\mathscr{M}'$ with parameters $\mathbf{a}$ and $\mathbf{b}$ are precisely the equivariant functions which are*

(1) *multiplicative discrete, if* $\mathbf{a} = \mathbf{b} = \mathbf{0}$;
(2) *primal elementary, if* $\mathbf{a} \neq \mathbf{0} = \mathbf{b}$;
(3) *dual elementary, if* $\mathbf{a} = \mathbf{0} \neq \mathbf{b}$;

(4) *normal, nil, paranil, pointlike* (*with distinguished point constituting a singleton color class*), *planar pairlike* (*with distinguished points constituting two singleton color classes*), *or collinear pairlike* (*with distinguished points constituting two singleton or one doubleton color class*), *if* $\mathbf{a}$ *and* $\mathbf{b} \neq \mathbf{0}$ *and if for any triple* $\{e_1, e_2, e_3\} \subseteq U'$, *there are matroids* $(f_1 f_2 f_3)_2$ *and* $(g_1 g_2 g_3)_1 \in \mathscr{M}'$ *with* $e_i \equiv f_i \equiv g_i$ *for* $i = 1, 2, 3$.

Proof. Let $\mathscr{M}''$ be the class of matroids M'' in U' such that $M'' \equiv M'$ for some $M' \in \mathscr{M}'$, and define $F(M'') = F(M')$. This extends F to $\mathscr{M}''$ as a strong Tutte function if F was a strong Tutte equivariant of $\mathscr{M}'$, because a colored isomorphism θ of M'' to M' induces a bijection of $\mathscr{M}(M'')$ with $\mathscr{M}(M')$ which preserves colors, parameters, function values, and the minor relationship. Now $\mathscr{M}''$ satisfies the conditions of Theorem 6.1. Theorem 6.2 follows. □

Proof of Theorem 2.2. We *reify* the colors, which means we construct a colored universe $U' = U \times C$, equivariant parameters $\mathbf{a}', \mathbf{b}' \in K^{U'}$ given by $\mathbf{p}'_{(e,c)} = \mathbf{p}_c$, and a certain new minor-closed class $\mathscr{M}'$. To define $\mathscr{M}'$ we need to construct from $(M, \kappa) \in [\mathscr{M}, C]$ a matroid M_κ in U'. The point set is $E(M_\kappa) = \{(e, \kappa(e)) : e \in M\}$ and the matroid M_κ is chosen so that projection on the first coordinate is an isomorphism $M_\kappa \to M$. Then $\mathscr{M}' = \{M_\kappa : (M, \kappa) \in [\mathscr{M}, C]\}$. Finally we define $F'(M_\kappa) = F(M, \kappa)$. Now apply Theorem 6.2 with F' as the function and $\mathbf{a}', \mathbf{b}'$ as the parameters. This gives Theorem 2.2. There are no pairlike or pointlike examples in Theorem 2.2 because every color class has at least three members. □

7. Basis expansions: the Tutte polynomial

An *ordered matroid* (M, O) is a matroid M together with a linear ordering O of its point set (or of a larger set). Given a fixed basis B we call a point e *internally active* [or, *inactive*] with respect to B if $e \in B$ and it is [or, is not] the largest element of the unique cocircuit in $B^c \cup \{e\}$. We call e *externally active* [or, *inactive*] with respect to B if $e \notin B$ and e is [or, is not] the largest element in the unique circuit in $B \cup \{e\}$. (The definitions of internal and external activity in graphs are due to Tutte [12]; they were extended to matroids by Crapo [1].) Let

$$B_+ = \{e \in B : e \text{ is internally active}\}, \qquad B_- = B \backslash B_+,$$
$$B^+ = \{e \notin B : e \text{ is externally active}\}, \qquad B^- = B^c \backslash B^+.$$

Now suppose parameters $\mathbf{a}, \mathbf{b} \in K^{E(M)}$ and variables $\mathbf{x}, \mathbf{y} \in K^{E(M)}$ are given. The (*ordered*) *parametrized Tutte polynomial* of (M, O) is

$$t_{M,O}(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y}) = \sum_B x(B_+) b(B_-) y(B^+) a(B^-),$$

summed over all bases B of M. (Here $z(S)$ denotes $\prod_{e \in S} z_e$.). If for particular values $(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y})$ the value of $t_{M,O}(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y})$ is independent of the ordering, we call this value the *parametrized Tutte polynomial* of M and write $t_M(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y})$ for the common value. In these circumstances we also say $t_M(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y})$ is *well defined*.

Of course t_M is not really a polynomial in $\mathbf{x}$ and $\mathbf{y}$ since their permitted values are constrained by the need for t_M to be well defined. However, when appropriate choices of $\mathbf{x}$ and $\mathbf{y}$ are made, t_M does become a polynomial; see the concluding remark of this section.

The main results about Tutte polynomials show that in a sense they are universal strong Tutte functions.

Theorem 7.1. *Given a set U, parameters $\mathbf{a}, \mathbf{b} \in K^U$, and variables $\mathbf{x}, \mathbf{y} \in K^U$, then $t_M(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y})$ is a well-defined strong Tutte function on $\mathscr{M}[\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y}]$ with parameters $\mathbf{a}$ and $\mathbf{b}$. Furthermore, if $\mathscr{M}$ is any minor-closed subclass of $\mathscr{M}(U)$ on which $t_M(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y})$ is well defined, then $\mathscr{M} \subseteq \mathscr{M}[\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y}]$.*

Theorem 7.2. *Given a strong Tutte function F on a minor-closed class $\mathscr{M}$ of matroids in a set U, let $\mathbf{x}$ and $\mathbf{y}$ be its point values and $\mathbf{a}, \mathbf{b}$ its parameters. Then $t_M(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y})$ is well defined and equals $F(M)$ for every $M \in \mathscr{M}$, unless F is identically zero* (*with* $F(\varnothing) = 0$).

Before proving these theorems we mention an orthogonal duality property of the Tutte polynomial. Let B be a basis of M. Its corresponding basis of $M^\perp$ is B^c. Calculating B_+, etc., in M and $(B^c)_+$, etc., in $M^\perp$, we see that $(B^c)_+ = B^+$, $(B^c)_- = B^-$, $(B^c)^+ = B_+$, and $(B^c)^- = B_-$. Therefore

$$t_{M,O}(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y}) = t_{M^\perp,O}(\mathbf{b}, \mathbf{a}; \mathbf{y}, \mathbf{x}). \tag{7.1}$$

We need some notation and a lemma. Let us write t_M as an abbreviation for $t_M(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y})$, $t_{M,O}$ for $t_{M,O}(\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y})$, etc. We shall take M to have point set $E(M) = \{e_1, e_2, \dots, e_n\}$, the points numbered in increasing order according to O. We write a_i for a_{e_i}, etc.

Lemma 7.3. *We have*

$$t_{M,O} = t_{M_1,O}t_{M_2,O} \quad \text{if } M = M_1 \oplus M_2$$

and

$$t_{M,O} = a_1 t_{M\backslash e_1,O} + b_1 t_{M/e_1,O} \quad \text{if } e_1 \text{ is not a separator of } M.$$

Proof. The first formula is obvious.

For the second formula we make some observations about a fixed nonseparating point e_i and basis B of M. Suppose $e_i \notin B$. A point $e_j \notin B$, other than e_i, has the same state of activity (with respect to B) in $M\backslash e_i$ as it does in M. A point $e_j \in B$ is active in M precisely when it is the largest element in $[\mathrm{clos}_M(B\backslash e_j)]^c$; it is active in $M\backslash e_i$ precisely when it is the largest element in $[\mathrm{clos}_{M\backslash e_i}(B\backslash e_j)]^c = [\mathrm{clos}_M(B\backslash e_j)]^c\backslash e_i$. Provided $j > i$, e_j is the largest element in both sets or in neither. Taking $i = 1$ we get

$$t_{M\backslash e_1,O} = \sum_{B \not\ni e_1} x(B_+)b(B_-)y(B^+\backslash e_1)a(B^-\backslash e_1), \tag{7.2}$$

where the calculations on the right are done in M with B ranging over bases of M not containing e_1.

If $e_i \in B$ we look at $M^\perp$ and its basis B^c. Applying (7.2) to $M^\perp\backslash e_1$ yields

$$t_{M^\perp\backslash e_1,O}(\mathbf{b}, \mathbf{a}; \mathbf{y}, \mathbf{x}) = \sum_{B^c \not\ni e_1} y((B^c)_+)a((B^c)_-)x((B^c)^+\backslash e_1)b((B^c)^-\backslash e_1),$$

summed over bases B^c of $M^\perp$ which do not contain e_1. Dualizing by means of (7.1) and the argument preceding it gives

$$t_{M/e_1,O} = \sum_{B\ni e_1} y(B^+)a(B^-)x(B_+\backslash e_1)b(B_-\backslash e_1), \tag{7.3}$$

summed over bases B of M which do contain e_1.

Now we observe that

$$\begin{aligned} t_{M,O} &= a_1 \sum_{B\not\ni e_1} x(B_+)b(B_-)y(B^+\backslash e_1)a(B^-\backslash e_1) \\ &\quad + b_1 \sum_{B\ni e_1} x(B_+\backslash e_1)b(B_-\backslash e_1)y(B^+)a(B^-) \\ &= a_1 t_{M\backslash e_1,O} + b_1 t_{M/e_1,O} \end{aligned}$$

by (7.2) and (7.3). □

The proof shows that we cannot in general expect $t_{M,O}$ to be linear with respect to other points than e_1.

Proof of Theorem 7.1. Lemma 4.1, equation (5.3), and Lemma 7.3 imply that t_M is well defined on $\mathscr{M}_{(3)}[\mathbf{p};\mathbf{q}]$. Lemma 7.3 and Corollary 3.3 imply that $t_{M,O}$ is a strong Tutte function on $\overline{\mathscr{M}_{(3)}[\mathbf{p};\mathbf{q}]}$, which equals $\mathscr{M}[\mathbf{p};\mathbf{q}]$ by (3.4). This is true equally well of $t_{M,O'}$ for any other ordering O'; hence $t_{M,O}$ and $t_{M,O'}$ are identical strong Tutte functions on $\mathscr{M}[\mathbf{p};\mathbf{q}]$. So t_M is a well-defined strong Tutte function on $\mathscr{M}[\mathbf{p};\mathbf{q}]$.

If t_M is well defined on a minor-closed class $\mathscr{M}$, it is multiplicative and linear on $\mathscr{M}$ by Lemma 7.3. Thus $\mathscr{M} \subseteq \mathscr{M}[\mathbf{p};\mathbf{q}]$ by Theorem 1.1. □

Proof of Theorem 7.2. Since $\mathscr{M} \subseteq \mathscr{M}[\mathbf{p};\mathbf{q}]$, t_M is a well-defined strong Tutte function on $\mathscr{M}$. So on $\mathscr{M}$ it is identical to F. □

Example 7.1. To show that the function t_M can be well defined even though $M \notin \mathscr{M}[\mathbf{a},\mathbf{b};\mathbf{x},\mathbf{y}]$ (but as we saw, M cannot be a *minimal* nonmember of $\mathscr{M}[\mathbf{a},\mathbf{b};\mathbf{x},\mathbf{y}]$) we give the example of $M = (e_1e_2e_3)_1$, a three-point cocircuit. We take $a_1 = a_2 = a_3 = 0$, $y_2 = y_3 = 0$, but $y_1, b_2, b_3 \neq 0$. We indicate the ordering of points by the subscript sequence ijk. We have $t_{M,ijk} = b_i y_j y_k = 0$ for all orderings, but

$$t_{(ij)_1,ij} = b_i y_j = \begin{cases} 0 & \text{if } j = 2 \text{ or } 3, \\ b_i y_1 \neq 0 & \text{if } j = 1, \end{cases}$$

so $t_{(12)_1}$ and $t_{(13)_1}$ are not well defined even though t_M is.

What goes wrong with t_M must happen at the bottom or not at all, according to the next result.

Corollary 7.4. *If* $\mathbf{a},\mathbf{b},\mathbf{x},\mathbf{y} \in K^U$ *are such that* $t_M(\mathbf{a},\mathbf{b};\mathbf{x},\mathbf{y})$ *is well defined for all digons, triangles, and triads in* U, *then* $t_M(\mathbf{x},\mathbf{y};\mathbf{a},\mathbf{b})$ *is a well-defined strong Tutte function on* $\mathscr{M}(U)$.

Proof. Under the hypothesis, $\mathscr{M}[\mathbf{a},\mathbf{b};\mathbf{x},\mathbf{y}] = \mathscr{M}(U)$. □

Theorem 7.2 implies formulas, which the reader can easily supply, expressing the Tutte "polynomial" as a polynomial function of the parameters, the other arbitrarily chosen quantities, and possibly 0, in each global example of §2. For example, the nil function $t_M(\mathbf{a}, \mathbf{b}; \mathbf{0}, \mathbf{0})$ is the polynomial $0^{|E(M)|}$.

8. Scaling

Pointwise scaling means transforming a function $F : \mathscr{M} \to K$ to F' defined by

$$F'(M) = F(M) \prod_{e \in M} \gamma_e, \tag{8.1}$$

where $\gamma \in (K^*)^U$. *Cross scaling* means transforming F to F'' defined by

$$F''(M) = \alpha^{\operatorname{rk} M} \beta^{\operatorname{nul} M} F(M), \tag{8.2}$$

where $\alpha, \beta \in K^*$. The names are explained by Proposition 8.1. For $\gamma, \boldsymbol{\pi} \in K^U$, let $\gamma \circ \boldsymbol{\pi}$ denote the pointwise product $(\gamma_e \pi_e : e \in U)$.

Proposition 8.1. *The function obtained from a strong Tutte function $F = F[\mathbf{a}, \mathbf{b}; \mathbf{x}, \mathbf{y}]$ through pointwise scaling by $\gamma \in (K^*)^U$ is the strong Tutte function $F' = F[\gamma \circ \mathbf{a}, \gamma \circ \mathbf{b}; \gamma \circ \mathbf{x}, \gamma \circ \mathbf{y}]$ on the same domain.*

The function obtained from F through cross scaling by α and β is the strong Tutte function $F'' = F[\alpha\mathbf{a}, \beta\mathbf{b}; \beta\mathbf{x}, \alpha\mathbf{y}]$ on the same domain.

Proof. The first part is obvious from the definitions. Both parts are obvious from the Tutte polynomial. □

Let us call strong Tutte functions *scaling equivalent* if they are related by pointwise and cross scaling. Evidently they are then essentially similar. One can use scaling to simplify the great variety of Tutte functions. Pointwise scaling allows one to assume, say, that all $a_e = 1$ or 0. Up to scaling equivalence, all the parameter vectors of every collinear function belong to $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. A primal elementary function with all $a_e \neq 0$ scales to

$$F(M) = u^{\operatorname{rk} M} \prod_{e \in E_0(M)} y_e,$$

a dual elementary function with all $b_e \neq 0$ scales to

$$F(M) = v^{\operatorname{nul} M} \prod_{e \in E_1(M)} x_e,$$

and any normal collinear function scales to the ordinary Tutte polynomial, $t_M(x, y) = R_M(x-1, y-1)$. However, scaling does not seem to help much with the most general type: normal functions whose parameter vectors are noncollinear.

9. Permutation and conjugation

Here we examine transformations of strong Tutte functions that are more substantial than scaling: permutations, which are induced by permuting the point universe, and conjugations, induced by dualizing and permuting. We are interested particularly in the functions which equal their own transforms; this will enable us in §11 to cast some new light on Kauffman's "Tutte polynomial of a signed graph" and his recursive link-diagram polynomial.

A *permutation* σ of U acts on functions of U, like K^U and $(K^U)^2$, and on $\mathscr{M}(U)$ in the obvious ways, e.g., $\gamma \in K^U$ becomes γ^σ given by $(\gamma^\sigma)_e = \gamma_{e^\sigma}$, and M becomes M^σ defined by $E(M^\sigma) = E(M)^\sigma$ and $\mathrm{rk}_{M^\sigma}(S^\sigma) = \mathrm{rk}_M(S)$. For $\mathscr{M} \subseteq \mathscr{M}(U)$ let $\mathscr{M}^\sigma = \{M^\sigma : M \in \mathscr{M}\}$. If F is a function on $\mathscr{M}$, F^σ is the function on $\mathscr{M}^\sigma$ given by $F^\sigma(M^\sigma) = F(M)$.

Duality acts on $\mathscr{M}$ by the rule $\mathscr{M}^\perp = \{M^\perp : M \in \mathscr{M}\}$, on F by the definition $F^\perp(M) = F(M^\perp)$, and on K^2 and $(K^U)^2$ by interchanging coordinates, i.e., $p = (a, b) \mapsto p^* = (b, a)$ and $\mathbf{p} = (\mathbf{a}, \mathbf{b}) \mapsto \mathbf{p}^* = (\mathbf{b}, \mathbf{a})$. Combining duality with a permutation σ yields the *σ-conjugate*: $M^{\perp\sigma}$, $\mathbf{p}^{*\sigma}$, or $F^{*\sigma}$.

The *supersymmetric group* of U, $\mathfrak{S}^*(U)$, consists of all permutations and conjugations of $\mathscr{M}(U)$. Its permutation part $\mathfrak{S}^0(U)$ is the symmetric group $\mathfrak{S}(U)$; its conjugation part is $\mathfrak{S}^1(U)$. The *supersymmetric group of $\mathscr{M} \subseteq \mathscr{M}(U)$*, the subgroup leaving $\mathscr{M}$ invariant, is written $\mathfrak{S}^*(\mathscr{M})$. An *action* of a group $\mathfrak{G}$ on $\mathscr{M}$ is a homomorphism $\rho : \mathfrak{G} \to \mathfrak{S}^*(\mathscr{M})$; we define $\mathfrak{G}^i = \rho^{-1}(\mathfrak{S}^i(\mathscr{M}))$ and we call $\sigma \in \mathfrak{G}$ a *permutation* or *conjugation* according as $\rho(\sigma)$ is one or the other. We normally suppress the symbol ρ. For an object X acted upon by $\mathfrak{G}$ and for $\mathfrak{S} \subseteq \mathfrak{G}$, $X\mathfrak{S}$ denotes $\{X^\sigma : \sigma \in \mathfrak{S}\}$. An object or set is, as usual, *invariant under* $\mathfrak{S}$ if it equals its image under the action of $\mathfrak{S}$, e.g., if $X\mathfrak{S} = X$; it is *self-conjugate under* $\mathfrak{S}$ if $\mathfrak{S}$ consists of permutations and it is invariant under $\{\perp\sigma : \sigma \in \mathfrak{S}\}$. We say *self-dual* for self-conjugate when $\mathfrak{S} = \{\text{identity}\}$. As is customary we shorten $\{\sigma\}$-invariant to σ-invariant, and so forth. A strong Tutte function $F = F[\mathbf{p}; \mathbf{q}]$ is *strongly* invariant (or, self-conjugate) if both F and its associated parameter sequence $\mathbf{p}$ are invariant (or, self-conjugate).

The action of $\mathfrak{G}$ is *odd* if there is a conjugation $\sigma \in \mathfrak{G}^1$ with a fixed point; otherwise it is *even.*

Lemma 9.1. *An action is even if and only if $e\mathfrak{G}^0 \cap e\mathfrak{G}^1 = \varnothing$ for every point e.* □

Most interesting perhaps is the case where $\mathfrak{G}$ is a cyclic subgroup $\langle\hat\sigma\rangle$ of $\mathfrak{S}^*(U)$. The action of $\langle\hat\sigma\rangle$ is odd if and only if $\hat\sigma$ is a conjugation whose permutation part has an odd cycle.

Suppose $\mathfrak{G}$ acts on $\mathscr{M}(U)$. The obvious way to get a strong Tutte function which is $\mathfrak{G}$-invariant is to take $F = F[\mathbf{p}; \mathbf{q}]$, where $\mathbf{p}$ and $\mathbf{q}$ are $\mathfrak{G}$-invariant. Not every example is of this type: consider for example a nil function with parameters that are not invariant; or more substantially $F(M) = \prod\{a_e + b_e : e \in M\} = R_M(\mathbf{a}, \mathbf{b}; 1, 1)$, where $\mathbf{a}+\mathbf{b}$ is $\mathfrak{G}$-invariant ($\mathfrak{G}$ being a permutation group); here the parameters can be varied as long as each sum $a_e + b_e$ remains constant. But $\mathfrak{G}$-invariant parameters do exist in both cases, with some exceptions: in the latter example for instance we may take $a'_e = b'_e = \frac{1}{2}(a_e + b_e)$ to get parameters that are both $\mathfrak{G}$-invariant and self-dual, except when $\mathrm{char}\, K = 2$ and $a_e + b_e \neq 0$. This behavior is typical.

Theorem 9.2. *A strong Tutte function of matroids which is $\mathfrak{G}$-invariant (where $\mathfrak{G}$ acts on $\mathscr{M}(U)$) always has a parameter sequence that makes it strongly $\mathfrak{G}$-invariant—with the possible exception, when* $\mathrm{char}\, K = 2$ *and $\mathfrak{G}$ has odd action, of a $\mathfrak{G}$-self-conjugate function which is degenerate or nonglobal.*

The proof depends in the first place on finding out how the parameters which may be associated with a given strong Tutte function F (are *feasible* for F)

are constrained by the values of F. For $e \in U$, let

$$\mathscr{M}_e = \{M \in \mathscr{M} : e \in E_*(M)\}$$

and put $F(M, e) = (F(M\backslash e), F(M/e)) \in K^2$. The constraints on p_e may be written as

$$p_e \cdot F(M, e) = F(M) \quad \text{for all } M \in \mathscr{M}_e. \tag{9.1}$$

Thus the set $S(e)$ of feasible parameters p_e forms an affine subspace of K^2 and the choices of parameter vectors p_e for different points e are independent.

Let F be $\mathfrak{G}$-invariant. We may assume $\mathfrak{G} \subseteq \mathfrak{S}^*(U)$ without loss of generality. We have $\mathscr{M}_{e^\sigma} = (\mathscr{M}_e)^\sigma$, $F(M^\sigma, e^\sigma) = F(M, e)$, and $S(e^\sigma) = S(e)$ for every $\sigma \in \mathfrak{G}^0$. Let $[e]$ denote the orbit of $\mathfrak{G}$ which contains e.

Let us look first at the case $\mathfrak{G} \subseteq \mathfrak{S}(U)$. Pick one fixed element $d_{[e]}$ in each orbit $[e]$ and define $\mathbf{p}' \in (K^U)^2$ by $p'_e = p_{d_{[e]}}$. Then $\mathbf{p}'$ is a $\mathfrak{G}$-invariant feasible parameter sequence for F, as required.

Suppose $\mathfrak{G} \not\subseteq \mathfrak{S}(U)$. For a conjugation $\hat{\sigma} = \perp \circ \sigma$, from $F^{\hat{\sigma}} = F$ it follows that $\mathscr{M}^{\hat{\sigma}} = \mathscr{M}$, $\mathscr{M}_{e^\sigma} = (\mathscr{M}^{\hat{\sigma}})_{e^\sigma} = (\mathscr{M}_e)^{\hat{\sigma}}$, $F(M^{\hat{\sigma}}) = F^{\hat{\sigma}}(M^{\hat{\sigma}}) = F(M)$, $F(M^{\hat{\sigma}}, e^\sigma) = F^{\hat{\sigma}}(M^{\hat{\sigma}}, e^\sigma) = F(M, \sigma)^*$, and hence

$$\begin{aligned} S(e^\sigma) &= \{p \in K^2 : p \cdot F(M^{\hat{\sigma}}, e^\sigma) = F(M^{\hat{\sigma}}) \ \forall M^{\hat{\sigma}} \in \mathscr{M}_{e^\sigma} = (\mathscr{M}_e)^{\hat{\sigma}}\} \\ &= \{p \in K^2 : p \cdot F(M, e)^* = F(M) \ \forall M \in \mathscr{M}_e\}, \end{aligned}$$

so that $S(e^\sigma) = S(e)^*$, where S^* denotes $\{p^* : p \in S\}$. Define

$$p'_e = \begin{cases} p_{d_{[e]}} & \text{if } e \in d_{[e]}\mathfrak{G}^0, \\ p^*_{d_{[e]}} & \text{if } e \in d_{[e]}\mathfrak{G}^1, \end{cases} \tag{9.2a}$$

if the action of $\mathfrak{G}$ on $[e]$ is even.

If the action of $\mathfrak{G}$ on $[e]$ is odd, let $\tau \in \mathfrak{G}^1$ have fixed point e. It follows that $S(e) = S(e^\tau) = S(e)^*$. Thus $p^*_e \in S(e)$ and, as long as $\operatorname{char} K \neq 2$, we can take

$$p'_e = \tfrac{1}{2}(p_e + p^*_e) \tag{9.2b}$$

to get self-conjugate parameters.

To solve the case of characteristic 2 we need a deeper analysis. We can restate $S(e) = S(e)^*$ as self-duality of the constraints on p_e : a constraint $p_e \cdot F(M, e) = F(M)$ implies $p_e \cdot F(M, e)^* = F(M)$ is also a constraint. Subtracting, we get

$$p_e \cdot [F(M, e) - F(M, e)^*] = 0.$$

So if $F(M\backslash e) \neq F(M/e)$ for even one $(M, e) \in \mathscr{M}_e$, we have $p_e \cdot (1, -1) = 0$, whose general solution is $p_e = (t, t)$. Therefore p_e is self-dual, and one can take $p'_e = p_{d_{[e]}}$.

Let U_1 be the set of all $e \in U$ such that $\mathfrak{G}$ acts oddly on $[e]$ and

$$F(M\backslash e) = F(M/e) \quad \text{for all } M \in \mathscr{M}_e, \tag{9.3}$$

and let $U_0 = \{e \in U_1 : F(M\backslash e) = 0 \ \ \forall M \in \mathscr{M}_e\}$. Each U_i is a union of orbits. The unsolved case is where U_1 is nonvoid. Assuming that F is global and has no $\mathfrak{G}$-self-conjugate parameter sequence and that $\operatorname{char} K = 2$, we prove that $U_1 \neq \varnothing$ implies F is degenerate. We may as well assume at the outset F is normal.

Say $e \in U_1$. Taking $M = (ef)_1$ in (9.3), we deduce

$$x_f = y_f \quad \text{for all } f \neq e. \tag{9.4}$$

If $e \in U_0$, then $q_f = 0$ for all $f \neq e$. Hence F is nil or pointlike, degenerate either way. So we may assume $U_0 = \varnothing$.

We show $U_1 \cap U^* \neq \varnothing$. Suppose on the contrary that $p_e = 0$ for all $e \in U_1$. Then $\mathbf{p}$ is self-conjugate on U_1 and (as in (9.2)) can be made so on $U \backslash U_1$, contradicting one of the hypotheses on F. So in fact there is a point $e \in U_1 \cap U^*$.

By applying (9.4) to e we infer $a_f(u-1) = b_f(v-1)$ for all $f \neq e$. If $u = v = 1$, F is degenerate; otherwise we have $p_f = \pi_f p$ if $f \neq e$, for some $p = (a, b) \in K^2$ and $\boldsymbol{\pi} \in K^{U \backslash e}$. Consequently $x_f = \pi_f x$ if $f \neq e$, where $x = au + b = a + bv$. If $x = 0$, F is obviously degenerate; therefore we can assume $x \neq 0$.

Now taking $M = (efg)_2$ in (9.3) we see that $F((fg)_2) = F((fg)_1)$, in other words $x_f x_g = a_f x_g + b_f y_g$. This reduces to $\pi_f \pi_g x^2 = \pi_f \pi_g (a+b)x$. We can take $f, g \in U^* \backslash e$ (or else F would be degenerate, by Lemma 2.2); then $\pi_e \pi_f x \neq 0$ so $a + b = x$. It follows that $a_f + b_f = x_f = y_f$ $\forall f \neq e$, and thence we have by induction on $|E_*(M)|$ the formula

$$F(M) = \lambda_e \cdot \prod_{f \in M \backslash e} x_f, \quad \text{where } \lambda_e = \begin{cases} 1 & \text{if } e \notin M, \\ x_e & \text{if } e \in E_0(M) \cup E_1(M), \\ a_e + b_e & \text{if } e \in E_*(M). \end{cases} \tag{9.5}$$

The conclusion: F is degenerate. □

We have already seen that there do exist global degenerate examples of self-conjugate strong Tutte functions in characteristic 2 which have no self-conjugate parameters. I do not know whether there exist any nonglobal, nondegenerate examples. One would like to know, for instance, whether there are such functions with domain $\mathscr{M}(M_0)$, where M_0 is a self-dual matroid which is large enough to be nontrivial.

One can apply Theorem 9.2 to colored matroids by reifying the colors as in the proof of Theorem 2.2 (in §6). That is to say, we define an action of $\mathfrak{G}$ on C-colored matroids in $\mathscr{M}$ to be an action of $\mathfrak{G}$ on $[\mathscr{M}, C]$, or on $\mathscr{M}'$ (see §6).

Theorem 9.3. *A strong Tutte function of colored matroids which is $\mathfrak{G}$-invariant (where $\mathfrak{G}$ acts on C-colored matroids in $\mathscr{M}$) always has a parameter sequence that makes it strongly invariant—with the possible exception, when* $\operatorname{char} K = 2$ *and the action is odd, of a $\mathfrak{G}$-invariant function which is degenerate or not global.* □

In a typical application $\mathfrak{G}$ acts exclusively on the colors, not the point universe U (see §11 for the case of two colors).

10. Graphs

A *weak Tutte function of graphs* is a function F defined on (finite) graphs which satisfies

$$F(\Gamma) = a_e F(\Gamma \backslash e) + b_e F(\Gamma / e) \quad \text{if } e \in E'(\Gamma), \tag{L$_G$}$$

where $E'(\Gamma)$ is the set of nonloop edges of the graph Γ. A *strong* Tutte function of graphs also satisfies

$$(\mathrm{M_G}) \qquad F(\Gamma_1 \uplus \Gamma_2) = F(\Gamma_1)F(\Gamma_2),$$

where $\uplus$ means disjoint union. These differ slightly from the corresponding matroid postulates: $(\mathrm{L_G})$ is stronger than (L) since in the former e may be an isthmus, but $(\mathrm{M_G})$ is weaker than (M). In this respect we follow the original work on the subject [11], where Tutte studied functions of graphs he called W-functions and V-functions, which are weak and strong Tutte functions having the invariance property $\Gamma_1 \cong \Gamma_2 \Rightarrow F(\Gamma_1) = F(\Gamma_2)$ as well as $a_e = b_e = 1$ for all e. (In modern terminology one might call a V-function a *Tutte-Grothendieck*, or *TG, invariant of graphs.*)

10a. **Portable Tutte functions.** Two further properties a function F might have are K_1-invariance and portability of loops: the properties that $F(K_1)$ is independent of the exact vertex in the K_1 and that $F(\Gamma)$ is not altered if a loop is moved from one vertex to another. We shall investigate mainly such functions, which we call *portable.* A portable TG invariant is, it turns out, merely a suitably adjusted TG matroid invariant; to be precise, an evaluation of the dichromatic polynomial

$$Q_\Gamma(u, v) = \sum_{S \subseteq E(\Gamma)} u^{c(S)} v^{|S|-n+c(S)} = u^{c(S)} R_{G(\Gamma)}(u, v),$$

where $n = |V(\Gamma)|$, $c(S)$ is the number of components of $(V(\Gamma), S)$, and $G(\Gamma)$ denotes the "polygon" or "circuit" matroid of Γ, whose matroid circuits are the graph circuits. We are therefore led to suspect a similar connection for strong Tutte functions and indeed it is almost true (see Theorems 10.1 and 10.2).

We begin with some notation. Γ denotes a finite graph with vertex set $V = V(\Gamma)$, edge set $E = E(\Gamma)$, order $n = |V|$, and size $m = |E|$. E_* is the set of nonloop, nonisthmus edges. We assume all graphs are finite with $E \subseteq U$, a fixed universal edge set; we call them graphs *in* U. Given a K_1-invariant function F of graphs, we define $\lambda = F(K_1)$ and $\lambda_e = F(V_e)$, V_e being the one-vertex graph whose single edge is a loop e. We also write $\boldsymbol{\lambda} = (\lambda_e : e \in U)$. If F depends only on the matroid $G(\Gamma)$, it can be regarded as a function of graphic matroids.

Theorem 10.1. *Let F be a portable strong Tutte function of all graphs in U with parameters $\mathbf{a}$ and $\mathbf{b}$ and with $\lambda \neq 0$. Then $\lambda^{-c(\Gamma)}F(\Gamma)$ is a strong Tutte function of graphic matroids in U which satisfies $\mathbf{x} = \mathbf{a}\lambda + \mathbf{b}$.*

Proof. Let $F'(\Gamma) = \lambda^{-c(\Gamma)}F(\Gamma)$. We use induction on $m = |E|$. For $m = 0$ the result is obvious. So let $m \geq 1$ and take Γ with m edges. We assume that for all graphs Γ_1 having fewer edges, $F'(\Gamma_1)$ depends only on $G(\Gamma_1)$; abusing notation slightly we write $F'(G(\Gamma_1))$ for this value.

Suppose $e \in E_*$. Then $(\mathrm{L_G})$ implies $F'(\Gamma) = a_e F'(\Gamma \backslash e) + b_e F'(\Gamma/e) = a_e F'(G(\Gamma)\backslash e) + b_e F'(G(\Gamma)/e)$. So $G(\Gamma)$ determines $F'(\Gamma)$ and F' satisfies (L).

Suppose $e \in E$ is an isthmus and let $\Gamma \backslash e = \Gamma_1 \uplus \Gamma_2$, where e joins a component of Γ_1 to one of Γ_2. In this case $G(\Gamma\backslash e) = G(\Gamma/e) = G(\Gamma)\backslash e$, so $(\mathrm{L_G})$ yields

$$(10.1) \qquad F'(\Gamma) = (a_e\lambda + b_e)F'(G(\Gamma)\backslash e).$$

Evidently $F'(\Gamma)$ depends only on $G(\Gamma)$. If we choose $\Gamma = K_2$ with e as its edge, (10.1) yields $x_e = (a_e\lambda + b_e)F'(\varnothing)$, which is $a_e\lambda + b_e$ because $F'(\varnothing) = 1$. (The only other possible value, $F'(\varnothing) = 0$, implies $F' \equiv 0$; but then $\lambda = \lambda F'(G(K_1)) = 0$, contrary to hypothesis.) Thus $F'(\Gamma) = x_e F'(G(\Gamma)\backslash e)$, as required by (M).

Supppose e is a loop. Then

$$F'(\Gamma) = \lambda^{-c(\Gamma)-1}F(\Gamma \uplus K_1) = \lambda^{-c(\Gamma)-1}F((\Gamma\backslash e) \uplus V_e) = F'(\Gamma\backslash e)F'(V_e).$$

Now, $y_e = F'(V_e)$ by definition. So (M) is satisfied here too.

In every case $F'(\Gamma)$ depends only on $G(\Gamma)$. $F'(G(\Gamma))$ satisfies (L) for every possible e and (M) if there is a separating point of $G(\Gamma)$. So F' is a strong Tutte function on $\mathscr{M}_{(3)}(G(\Gamma))$ and, by Theorem 3.1, on $\mathscr{M}(G(\Gamma))$.

By induction, F' is a strong Tutte function of graphic matroids which has $\mathbf{x} = \mathbf{a}\lambda + \mathbf{b}$. □

The same calculations make the converse obvious.

Theorem 10.2. *Let F' be a strong Tutte function of all graphic matroids in U such that $\mathbf{x} = \mathbf{a}u + \mathbf{b}$ for some u. Then $F(\Gamma)$, defined as $u^{c(\Gamma)}F'(G(\Gamma))$, is a portable strong Tutte function of all graphs in U and satisfies $\lambda = u$.* □

For the classification theorem we need a lemma.

Lemma 10.3. *Let F be a K_1-invariant strong Tutte function of all graphs in U. Then $\mathbf{b} = \mathbf{0}$ or $\boldsymbol{\lambda} = \mathbf{a}\lambda + \mathbf{b}\mu$ for some scalar μ.*

Proof. Let us calculate F on the digon $K_2(ef)$, whose edges are e and f. From ($\mathrm{L_G}$) and ($\mathrm{M_G}$) we deduce

$$\begin{aligned} F(K_2(ef)) &= a_e a_f \lambda^2 + a_e b_f \lambda + b_e \lambda_f \\ &= a_f a_e \lambda^2 + a_f b_e \lambda + b_f \lambda_e. \end{aligned}$$

We conclude that

$$\begin{vmatrix} \lambda_e - a_e\lambda & \lambda_f - a_f\lambda \\ b_e & b_f \end{vmatrix} = 0,$$

from which the lemma follows. □

Now we define the *parametrized dichromatic polynomial* with indeterminates u and v:

$$\begin{aligned} Q_\Gamma(\mathbf{a}, \mathbf{b}; u, v) &= u^{c(\Gamma)} R_{G(\Gamma)}(\mathbf{a}, \mathbf{b}; u, v) \\ &= \sum_{S \subseteq E} u^{c(S)} v^{|S|-n+c(S)} \cdot \prod_{e \in S^c} a_e \cdot \prod_{e \in S} b_e. \end{aligned}$$

Here S^c denotes $E\backslash S$. Evaluating u and v gives a function $F(\Gamma) = Q_\Gamma(\mathbf{a}, \mathbf{b}; u, v)$, which we call *normal.* It is a portable strong Tutte function of graphs, by Theorem 10.2.

Versions of the parametrized dichromatic polynomial have appeared in the literature. The earliest I know of is the polynomial $Z(G)$ of Fortuin and Kasteleyn [3, §7], which equals $Q_\Gamma(\mathbf{a}, \mathbf{b}; u, 1)$ with the minor restriction $a_e + b_e = 1$. Traldi [10] defines a two-variable polynomial which amounts to $Q_\Gamma(\mathbf{1}, \mathbf{b}; u, v)$; this is scaling equivalent to our apparently more general polynomial as long as no $a_e = 0$.

I should mention that a chromatic formula for $Q_\Gamma(u, v)$ due to Tutte [11, Theorem X] generalizes to the parametrized polynomial. A *k-coloring* of Γ is a function $\psi : V \to \{1, 2, \dots, k\}$; it is *proper* if every edge is proper, where an edge whose endpoints are v_1 and v_2 is proper if $\psi(v_1) \neq \psi(v_2)$. (Hence a loop is improper.) The *chromatic polynomial* $\chi_\Gamma(k)$ counts the number of proper k-colorings of Γ. Let $I(\psi) = \{e \in E : e \text{ is improper}\}$. An edge set $A \subseteq E$ is *closed* if, whenever $e \in E$ has endpoints connected by a path in A, then $e \in A$. Evidently $I(\psi)$ is always closed in the polygon matroid.

Proposition 10.4. *For a nonnegative integer k the parametrized dichromatic polynomial is given by*

$$Q_\Gamma\left(\mathbf{a}, \mathbf{b}; \frac{k}{v}, v\right) = v^{-n} \sum_{\psi} \prod_{e \in I(\psi)} (a_e + b_e v) \cdot \prod_{e \notin I(\psi)} a_e$$

summed over all k-colorings ψ of Γ,

$$= v^{-n} \sum_{A} \prod_{e \in A} (a_e + b_e v) \cdot \prod_{e \notin A} a_e \cdot \chi_{\Gamma/A}(k)$$

summed over closed edge sets A.

Proof. The first formula follows by the reasoning of Tutte [11, Theorem X]. The second follows from the first upon replacing the sum over ψ by a double sum over closed sets A and then over colorings ψ such that $I(\psi) = A$. □

The first formula is a direct generalization of [11, Theorem X]. It was suggested to me by Kauffman's expression [6, p. 107] for his $Z(G)$ (which he calls the dichromatic polynomial, but which is slightly different from Tutte's and my dichromatic polynomial).

Recall that $y_e = \lambda^{-1} F(V_e)$.

Theorem 10.5. *A portable strong Tutte function F of all graphs in U with $\lambda \neq 0$ is either normal with $u = \lambda$, or has $\mathbf{b} = \mathbf{0}$ and is the adjusted primal elementary function*

$$F(\Gamma) = \lambda^n \prod_{\substack{e \in E \\ \text{nonloop}}} a_e \cdot \prod_{\substack{e \in E \\ \text{loop}}} y_e. \tag{10.2}$$

Conversely, any function F of these two types, even with $\lambda = 0$, is a portable strong Tutte function of all graphs in U.

Proof. Assume F is a portable strong Tutte function with $\lambda \neq 0$. $F(\varnothing) = 0$ would imply $\lambda = 0$, so $F(\varnothing) = 1$. When $\mathbf{b} \neq \mathbf{0}$, we see from the lemma that $y_e = \mathbf{a} + \mathbf{b}(\mu/\lambda)$ for some μ, so we take $u = \lambda$ and $v = \mu/\lambda$. Otherwise we use Theorem 10.1 directly.

The converse is clear. □

Theoretically speaking, the significance of Theorems 10.2 and 10.5 is that four apparently distinct properties of a graphic strong Tutte function F are almost equivalent if $\lambda \neq 0$. They are, in increasing order of apparent strength:

(10.3a) Portability.

(10.3b) $F(\Gamma)$ depends only on $G(\Gamma)$ and n.

(10.3c) $F(\Gamma) = \lambda^{c(\Gamma)} F'(G(\Gamma))$ for some matroidal strong Tutte function F'.

(10.3d) $F(\Gamma) = Q_\Gamma(\mathbf{a}, \mathbf{b}; u, v)$ for some u and v.

We have shown that (10.3a) implies (10.3c) when $\lambda \neq 0$ and implies (10.3d) with a few exceptions.

When $\lambda = 0$, those implications do not hold. Indeed the case $\lambda = 0$ is remarkably complicated. (Fortunately it is not very important.) We mention one fact without proof: If $\lambda = 0$, then $\lambda_d \neq 0$ for at most one $d \in U$, and if some $\lambda_d \neq 0$ then $b_e = 0$ for all $e \neq d$ and $F(\Gamma) = 0$ whenever $n > 1$ or $d \notin E$.

10b. **Edge Tutte functions.** Suppose we postulate (for use in §11) slightly different properties of a function: (L'_G), which is (L_G) restricted to nonisthmus edges e, and a graphical discrete multiplicativity

$$(\mathrm{DM_G}) \qquad F(\Gamma) = \prod_{\text{isthmus}} x_e \cdot \prod_{\text{loop}} y_e \quad \text{if } \Gamma \text{ is a forest with loops.}$$

Such a function is really matroidal. Recall that $G(\Gamma)$ is the polygon matroid of Γ.

Proposition 10.6. *A function F of all graphs in U which satisfies (L'_G) and $(\mathrm{DM_G})$ equals $F' \circ G$ for some global strong Tutte function F' of matroids in U.*

Proof. We show that $F(\Gamma)$ depends only on $G(\Gamma)$, by induction on the number of edges which are neither loops nor isthmi. When this number is zero the result is immediate from $(\mathrm{DM_G})$. Otherwise let $e \in E^*$. Then

$$F(\Gamma) = a_e F(\Gamma \backslash e) + b_e F(\Gamma / e) = a_e F'(G(\Gamma) \backslash e) + b_e F'(G(\Gamma)/e),$$

where F' is, inductively, well defined by the formula $F'(G(\Gamma_1)) = F(\Gamma_1)$ on graphic matroids with fewer nonloop, nonisthmus edges. Thus $F(\Gamma)$ itself is a function of $G(\Gamma)$ alone.

Now F' satisfies (L) and (DM) so Corollary 3.4 applies. □

10c. **Edge-colored graphs.** An *edge-C-colored graph* is a pair $\Sigma = (\Gamma, \kappa)$, where $\kappa : E \to C$ is an edge coloring. We assume C is fixed and each color $c \in C$ comes with parameters a_c and b_c. Define $\Sigma \backslash e = (\Gamma \backslash e, \kappa|_{E \backslash e})$ and $\Sigma / e = (\Gamma / e, \kappa|_{E \backslash e})$ for $e \in E$. A *weak Tutte function of edge-C-colored graphs in U* is a function F defined on edge-C-colored graphs with $E \subseteq U$ and satisfying

$$(\mathrm{L_{CG}}) \qquad F(\Sigma) = a_{\kappa(e)} F(\Sigma \backslash e) + b_{\kappa(e)} F(\Sigma / e) \quad \text{if } e \in E',$$

$(\mathrm{I_{CG}})$ $F(\Sigma_1) = F(\Sigma_2)$ if there is a color-preserving isomorphism $\Gamma_1 \cong \Gamma_2$;

it is *strong* if it also obeys

$$(\mathrm{M_{CG}}) \qquad F(\Sigma_1 \uplus \Sigma_2) = F(\Sigma_1) F(\Sigma_2).$$

We call F *portable* if its value is not changed by moving a loop from one vertex to another (without changing its color, of course). Again let λ denote the value of F on K_1 and let $y_c = \lambda^{-1}F(V_e, c)$ for $c \in C$, where (V_e, c) is a c-colored loop e. The (parametrized) *dichromatic polynomial* of an edge-colored graph $\Sigma = (\Gamma, \kappa)$ with parameters $\mathbf{a}, \mathbf{b} \in K^C$ is

$$\begin{aligned} Q_\Sigma(\mathbf{a}, \mathbf{b}; u, v) &= u^{c(\Gamma)} R_{(G(\Sigma), \kappa)}(\mathbf{a}, \mathbf{b}; u, v) \\ &= \sum_{S \subseteq E} u^{c(S)} v^{|S|-n+c(S)} \cdot \prod_{e \in S^c} a_{\kappa(e)} \cdot \prod_{e \in S} b_{\kappa(e)}. \end{aligned}$$

Any evaluation of $Q_\Sigma(\mathbf{a}, \mathbf{b}; u, v)$ is called *normal.*

Theorem 10.7. *Let F be a portable strong Tutte function, with $\lambda \neq 0$, of all edge-C-colored graphs in a set U. Then $F'(G(\Gamma), \kappa) = \lambda^{-c(\Gamma)} F(\Gamma, \kappa)$ is a well-defined global strong Tutte function of C-colored matroids in U, and F is either normal with $u = \lambda$, or has all $b_c = 0$ and is given by*

$$F(\Sigma) = \lambda^n \prod_{\text{nonloop}} a_{\kappa(e)} \cdot \prod_{\text{loop}} y_{\kappa(e)}.$$

Conversely, any function F of either of these types, even with $\lambda = 0$, is a strong Tutte function of all edge-C-colored graphs in U.

Proof. Directly from Theorem 10.5. □

There are obvious colored analogs of (L'_G) and (DM_G); call them $(\mathrm{L}'_\mathrm{CG})$ and $(\mathrm{DM}_\mathrm{CG})$. We call a function obeying them and (I_CG) a *strong edge Tutte function of edge-colored graphs.* We then have

Proposition 10.8. *A function F of all edge-C-colored graphs in U which obeys (I_CG), $(\mathrm{L}'_\mathrm{CG})$, and $(\mathrm{DM}_\mathrm{CG})$ has the form $F(\Gamma, \kappa) = F'(G(\Gamma), \kappa)$, where F' is a global strong Tutte function of C-colored matroids in U.* □

10d. **Two concluding notes.** First, our results are stated for functions of all graphs in U, or all graphic matroids, but they remain valid if the domains of the graph functions are restricted to a minor-closed class which includes all triangles and triple-parallel-edge graphs in U, when $|U| \geq 3$, and if the matroid functions are correspondingly restricted. Thus we could have said "all planar graphs in U" and "all graphic-cographic matroids in U," for example.

Secondly, we use terms like "conjugation" and "self-conjugate" just as for matroids (see §9) with the obvious modifications. Notably, in defining self-conjugacy of graph functions we require $F(\Gamma) = F(\Gamma^{*\sigma})$ (Γ^* being the dual graph) only when Γ is dualizable, that is to say, planar.

11. Kauffman's Tutte polynomial

We are finally in a position to fit Kauffman's "Tutte polynomial of a signed graph" into our system. A ± 1-*labelled graph*[1] is an edge-colored graph $\Sigma = (\Gamma, \kappa)$ where the color (or "label") set is $\{+1, -1\}$; then the coloring map is $\kappa : E \to \{+1, -1\}$. For $\varepsilon = \pm 1$ let $E_*(\varepsilon)$ be the set of nonloop, nonisthmus edges, and $m_0(\varepsilon)$ and $m_1(\varepsilon)$ the numbers of loops and isthmi, whose color

[1] Kauffman's "signed graph." I think it better to regard the color set $\{+1, -1\}$ as acted upon by but not identical with the sign group $\{+, -\}$, because the labels do not multiply in an interesting way. When they do, the phenomena are different.

is ε. We take it that when we dualize a ± 1-labelled graph we reverse the colors. We call this operation *conjugation* to avoid confusion with ordinary graphical duality. Of course it is a conjugation in the sense of §9. Conjugation arises naturally in the study of recursive invariants of link diagrams: the two ± 1-labelled graphs that are naturally associated with the two region types in a 2-coloring of the regions of the diagram are conjugate.

We shall call Kauffman's polynomial $T(\Sigma)$. Let A and B be indeterminates. His definition, from [4] or [5, p. 233], is recursive, with three parts:

$$(11.1)\qquad T(\Sigma) = \begin{cases} BT(\Sigma\backslash e) + AT(\Sigma/e) & \text{if } e \in E_*(+1), \\ AT(\Sigma\backslash e) + BT(\Sigma/e) & \text{if } e \in E_*(-1), \end{cases}$$

and

$$(11.2)\qquad T(\Sigma) = X^{m_1(+1)+m_0(-1)} Y^{m_0(+1)+m_1(-1)} \quad \text{if } \Sigma \text{ is a forest with loops},$$

where

$$(11.3)\qquad X = Bd + A \quad \text{and} \quad Y = B + Ad \quad \text{for an indeterminate } d.$$

These formulas evidently determine $T(\Sigma)$ as a function of A, B, and d, provided it is well defined.

For $S \subseteq E$, let $m(S, \varepsilon)$ be the number of ε-colored edges in S.

Proposition 11.1. *Given that A and B are indeterminates and* $T(\Sigma)$ *is not identically zero on nonnull signed graphs, equations* (11.1) *and* (11.2) *have the unique solution*

$$(11.4)\qquad \begin{aligned} T(\Sigma) &= R_{(G(\Gamma),\kappa)}(\mathbf{a}, \mathbf{b}; d, d) \\ &= \sum_{S \subseteq E} d^{|S|+2c(S)-n-c(\Gamma)} A^{m(S,+1)+m(S^c,-1)} B^{m(S^c,+1)+m(S,-1)} \end{aligned}$$

(*where* $a_{+1} = b_{-1} = B$ *and* $a_{-1} = b_{+1} = A$); *whence* (11.3) *follows.*

Proof. The postulated equations are special cases of ($\mathrm{L}'_{\mathrm{CG}}$) and ($\mathrm{DM}_{\mathrm{CG}}$) with

$$a_{+1} = b_{-1} = B, \quad b_{+1} = a_{-1} = A, \quad x_{+1} = y_{-1} = X, \quad y_{+1} = x_{-1} = Y.$$

We conclude from Proposition 10.8 that T is a global strong Tutte function of 2-colored matroids, restricted to graphic matroids, and from Theorem 2.2 and the fact that $p_{+1} = (B, A)$ and $p_{-1} = (A, B)$ are linearly independent (because A and B are indeterminates) that T is either nil or normal. The former is ruled out by hypothesis. The latter entails

$$X = x_{+1} = Bu + A, \quad X = y_{-1} = A + Bv,$$
$$Y = y_{+1} = B + Av, \quad Y = x_{-1} = Au + B.$$

Then $Bu + A = A + Bv$ implies $u = v$. Call their common value d. Since $\operatorname{rk} S = n - c(S)$ for any $S \subseteq E$, in particular cork S = rk $\Gamma-$ rk S = $c(S) - c(\Gamma)$, we have (11.4). □

Thus T is well defined and the hypotheses (11.3) are redundant if T is assumed not totally trivial. The maximal-forest expansion of $T(\Sigma)$ now becomes a consequence of our general theory (§7), since a basis of $G(\Gamma)$ is the same as a maximal forest in Γ, and so does the dichromatic formula

$$(11.5)\qquad T(\Sigma) = d^{-1} Q_\Sigma(\mathbf{a}, \mathbf{b}; d, d) \quad \text{if } \Sigma \text{ is connected and nonnull},$$

a restatement of (11.4). We also deduce that $T(\Sigma)$ is preserved by Whitney 2-isomorphism operations (see [14] or [8, §6.3]), since $G(\Sigma)$ is so preserved, by Whitney's theorem.

A more interesting characterization of $T(\Sigma)$ follows from §9. A function of ± 1-labelled graphs is *self-conjugate* if it takes the same value on conjugate graphs. The meaning of the next result is that $T(\Sigma)$ is the universal interesting self-conjugate strong Tutte function.

Theorem 11.2. *Kauffman's polynomial $T(\Sigma)$ is a self-conjugate, nondegenerate strong edge Tutte function of ± 1-labelled graphs. Any such function of ± 1-labelled graphs (or ± 1-labelled planar graphs) equals the function obtained by choosing appropriate values of A, B, and d in $T(\Sigma)$.*

Proof. Apply Proposition 10.8 first. Then by Theorem 9.3 self-conjugacy and nondegeneracy of the function imply it equals a strongly self-conjugate strong Tutte function. Then apply Theorem 2.2. □

The main significance of the theorem is that one cannot gain anything by allowing parameters that are not self-conjugate. This is not entirely obvious a priori. In the original domain of this subject, planar diagrams of links, it is natural to take only self-conjugate parameters (and consequently self-conjugate functions) because that is one way to make the defining recurrence (L'_{CG}) (cf. §10c) directly meaningful for link diagrams. (Kauffman discusses the diagrammatic approach in [5, §II, pp. 204–205].) However, once graphs are introduced it is natural to postulate only self-conjugacy of the function. One could hope that new self-conjugate functions would arise from non-self-conjugate parameters. But in fact they do not, by Theorem 11.2.

Kauffman's definition is not always as stated above. In [6, §IV] he adds the postulate

$$T(\Sigma_1 \uplus \Sigma_2) = kT(\Sigma_1)T(\Sigma_2), \tag{11.6}$$

with $k = d$. (To avoid nonsense Σ_1 and Σ_2 should be nonnull.) In [5, p. 235], (11.6) is added with $k = AX + BY$. If (11.6) is postulated, (11.2) must be restricted to connected graphs. Then obviously T has the form $k^{c(\Gamma)-1}F(G(\Gamma), \kappa)$, where F is a global strong Tutte function of colored matroids, necessarily (due to the choice of parameters) either normal with $u = v = d$ or nil. To get a Tutte-polynomial basis expansion of T for disconnected Σ, the basis expansion formula must be modified by inserting a corrective factor of $k^{c(\Gamma)-1}$.

Finally, some historical remarks. The dichromatic formula (11.5) for T and the conclusion that T is well defined were obtained, independently from each other and this work, by Traldi and Murasugi. Traldi used his general "weighted dichromatic polynomial," $Q_\Gamma(\mathbf{1}, \mathbf{b}; u, v)$, with weights $b_e = A/B$ if e is negative and $b_e = B/A$ if e is positive [10, §3]. Murasugi [7, Definition 2.1] defined a modified dichromatic polynomial for an edge-2-colored graph which is $y^{-1}Q_\Sigma(\mathbf{1}, \mathbf{b}; y, z)$ (or 0 if $\Sigma = \varnothing$) with $b_\varepsilon = x^\varepsilon$ and observed that (11.5) holds if $x = B/A$. Although neither author explicitly mentions that T depends only on the colored matroid $(G(\Gamma), \kappa)$, Murasugi does state that his polynomial is preserved by 2-isomorphism [7, p. 5], which by Whitney's theorem implies it is a colored-matroid invariant.

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